C. Handlebodies

index of handle

an n-dimensional k-handle is
$$h^k = D^k \times D^{n-k}$$

Set $\partial_- h^k = (\partial D^k) \times D^{n-k}$ attaching region

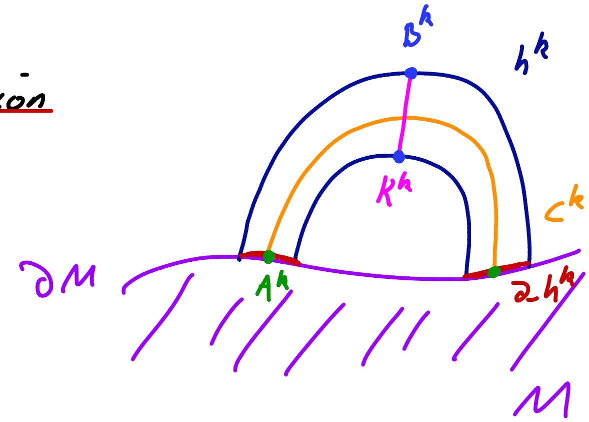
$\partial_+ h^k = D^k \times (\partial D^{n-k})$

$A^k = (\partial D^k) \times \{0\}$ attaching sphere

$C^k = D^k \times \{0\}$ core

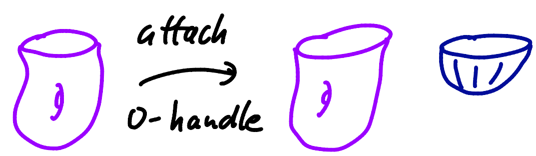
$K^k = \{0\} \times D^{n-k}$ co-core

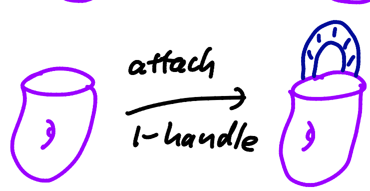
$B^k = \{0\} \times (\partial D^{n-k})$ belt sphere

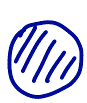
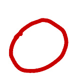


given an n-manifold M and an embedding
$$\phi : \partial_- h^k \to \partial M$$
we attach h^k to M by forming the identification space
$$M \amalg h^k \big/ (x \in \partial_- h^k) \sim (\phi(x) \in \partial M)$$

e.g. dimension 2:

$\underline{k=0}$:  $\partial_- h^0 = \emptyset$ so 

$\underline{k=1}$:  $\partial_- h^1 =$  so 

$\underline{k=2}$:  $\partial_- h^2 =$  so 

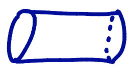
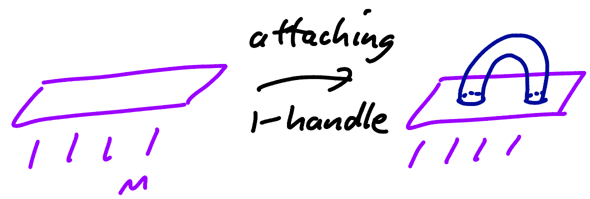
Remark:

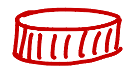
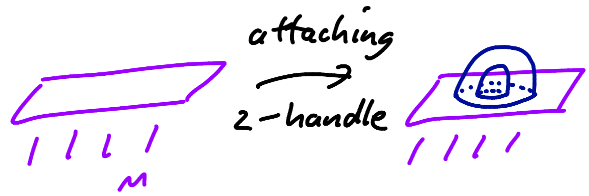
1) In all dimensions attaching a 0-handle is just taking disjoint union with D^n

2) In all dimensions n attaching an n-handle is just "capping off" an S^{n-1} boundary component.

dimension 3:

$\underline{h=0}$: ✓

$\underline{h=1}$:  $\partial_- h^1 =$  so 

$\underline{h=2}$:  $\partial_- h^2 =$  

$\underline{h=3}$: ✓

Remark: Note when a handle is attached one has a manifold with "corners" there is a standard way to smooth them out (see Wall "Differential Topology" or maybe in one of our afternoon talks)

exercises:

1) if $\phi_0, \phi_1 : \partial_- h^k \to \partial M$ are isotopic, then the result of attaching a handle to M via ϕ_0 is diffeomorphic to attaching a handle to M via ϕ_1

2) the isotopy class of $\phi : \partial_- h^k \to \partial M$ is determined by

   1) isotopy class of $\phi|_{A^k}$ $\quad (A^k = S^{k-1} \times \{0\})$

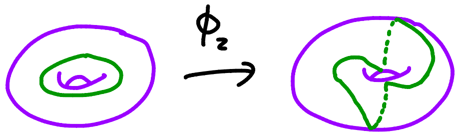
      (i.e. a S^{k-1} knot in ∂M)

   2) the "framing" of the normal bundle of $\phi(A^k)$

      i.e. an identification of $\nu(\phi(A^k))$ with $S^{k-1} \times D^{n-k}$

e.g. notice that $S^1 \times D^2$ has an integers worth of framings

$$S^1 \times D^2 \xrightarrow{\phi_n} S^1 \times D^2$$
$$(\phi, (r, \theta)) \longmapsto (\phi, (r, \theta + n\phi))$$



3) more generally show the framings on a k-dimensional sphere in Y^n is in one-to-one correspondence with $\pi_k(O(n-k))$

dim of normal bundle

so we see to attach an n-dimensional k-handle one must specify 1) an S^{k-1} knot in ∂M and

2) "elt" of $\pi_{k-1}(O(n-k))$

to really get such an element need a canonical "zero" framing

examples

1) 0-handles in an n-manifold

attaching sphere is $\emptyset$ and framing $\pi_{-1}(O(0)) = \emptyset$

so no choices! we saw this before attaching a 0-handle is just $\cup$ with B^n

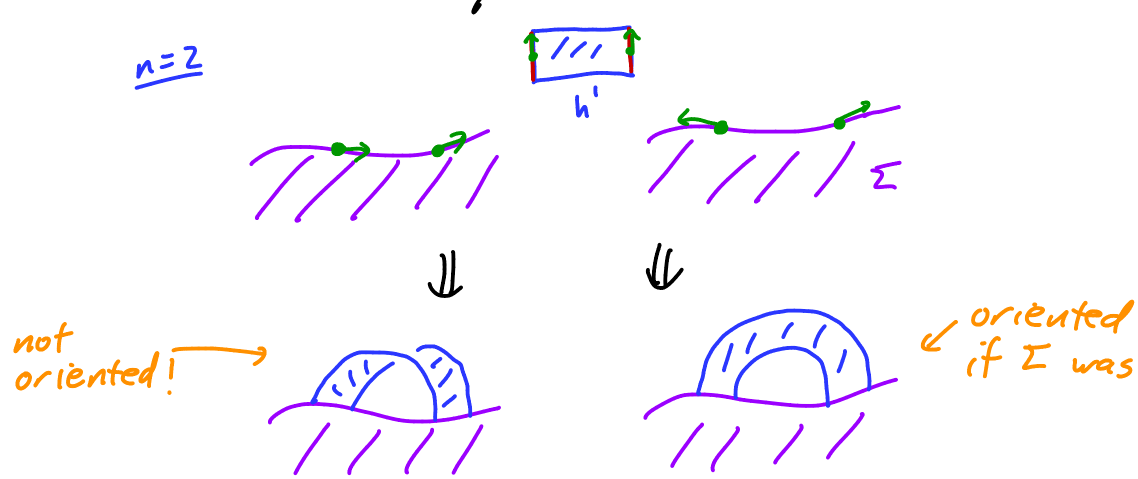
2) 1-handles in an n-manifold

attaching sphere is $S^0 = \cdot\ \cdot$

framing $\pi_0(O(n-1)) = \mathbb{Z}_2 \quad (n>1)$

so once you pick out 2 points to attach 1-handle there are 2 ways to attach the handle

$n=2$



this happens in general: when attaching a 1-handle once attaching sphere specified there is a <u>unique</u> way to attach a handle and preserve orientability

3) 2-handles in an n-manifold

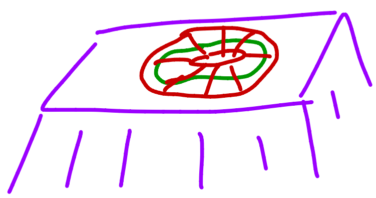
attaching sphere an S^1

framing element of $\pi_1(O(n-2)) = \begin{cases} \{e\} & n-2 = 0,1 \\ \mathbb{Z} & n-2 = 2 \\ \mathbb{Z}_2 & n-2 > 2 \end{cases}$

$n=2$

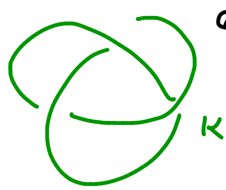
no choice once S^1 specified

$n=3$

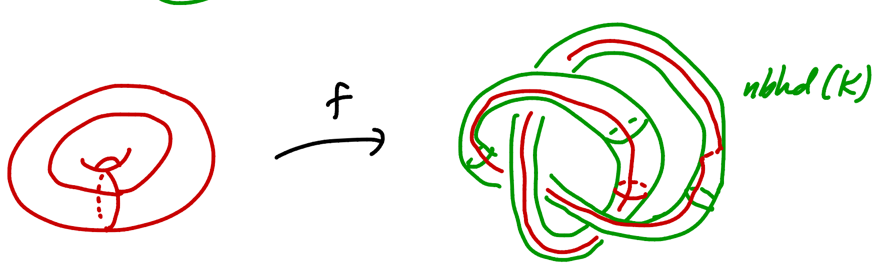


" "

$n=4$



attaching sphere a knot in 3-mfd $= \partial$(4-mfd)



get other "framings" by composing $f \circ \phi_n$

$\phi_n : S^1 \times D^2 \to S^1 \times D^2 : (\phi, r, \theta) \mapsto (\theta, r, \theta + n\phi)$

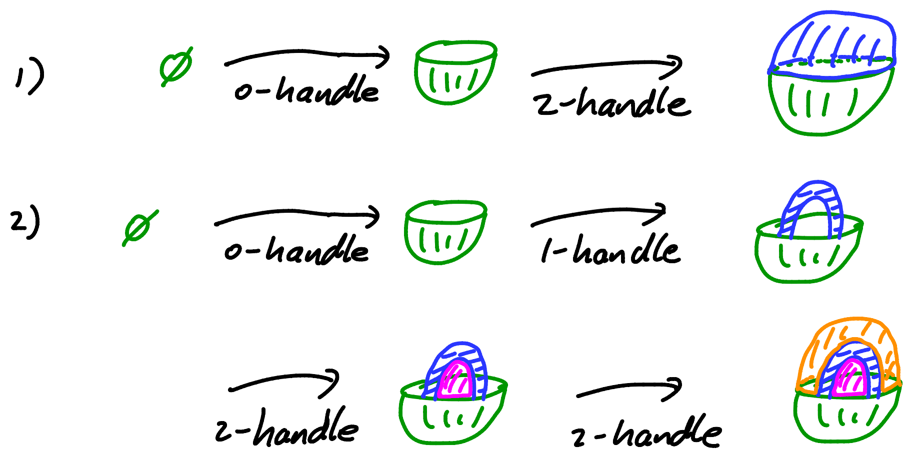
A <u>handle decomposition</u> of an n-manifold M is a sequence of manifolds $M_0, M_1, \ldots M_\ell$ such that

1) $M_0 = \emptyset$ and $M_\ell \cong M$
2) M_{i+1} is obtained from M_i by a k-handle attachment for some k

a handle decomposition of a cobordism M with $\partial_- M \neq \emptyset$ is the same except $M_0 \cong [0,\varepsilon] \times \partial_- M$

example:

handle decompositions of S^2



Th^m:

$\boxed{\text{Any smooth compact manifold has a handle decomposition}}$

This follows from the existence of Morse functions!

Main Th^m of Morse Theory:

let $f: M \to \mathbb{R}$ be a Morse function

I) if $[a,b]$ contains no critical values then

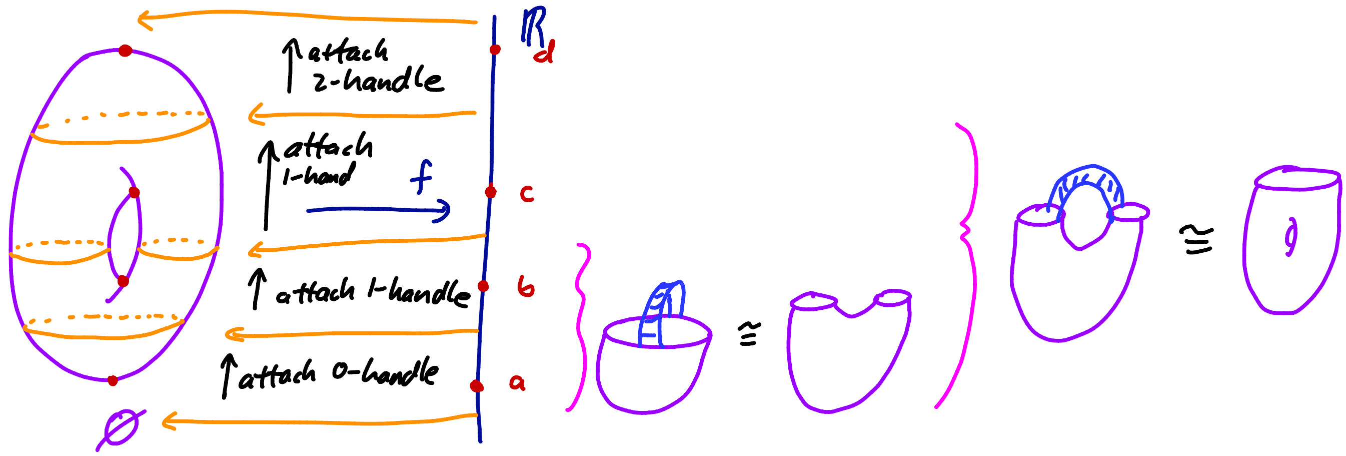
$$f^{-1}([a,b]) = f^{-1}(a) \times [a,b]$$

manifold since a regular value

II) if $\exists!$ critical point $p \in f^{-1}([a,b])$ of index k s.t. $f(p) \in (a,b)$ then

$f^{-1}([a,b])$ is obtained from $f^{-1}(a) \times [a, a+\varepsilon]$ by attaching a k-handle to $f^{-1}(a) \times \{a+\varepsilon\}$

example:



Remark: handle decomposition theorem clearly follows

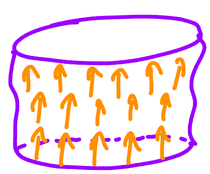
Idea of proof of "Main Th^{m}":

I) let $\Phi_t : M \to M$ be the (normalized) gradient flow of f

then
$$\begin{aligned} f^{-1}(a) \times [0, b-a] &\longrightarrow M \\ (p,t) &\longmapsto \Phi_t(p) \end{aligned}$$

is an embedding onto $f^{-1}([a,b])$

maybe say more in afternoon



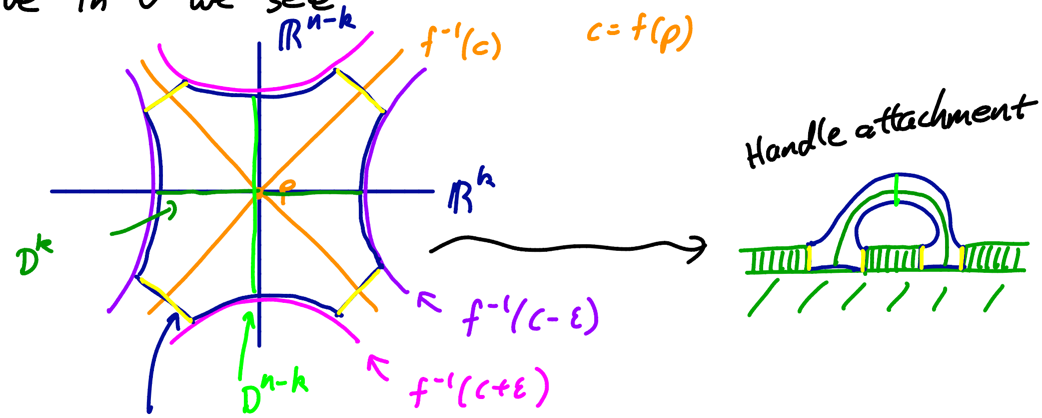
II) We start with a lemma

maybe prove in afternoon

lemma (Fundamental lemma of Morse theory):

> If p is a non-degenerate critical point of $f: M \to \mathbb{R}$ with index k then $\exists$ coordinates about p such that f takes the form
>
> $$f(x_1, \dots, x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

II) let U be nbhd about p where f has the form as in exercise 3 above in U we see



this is essentially a k-handle attachment

exercise: finish proof of II)

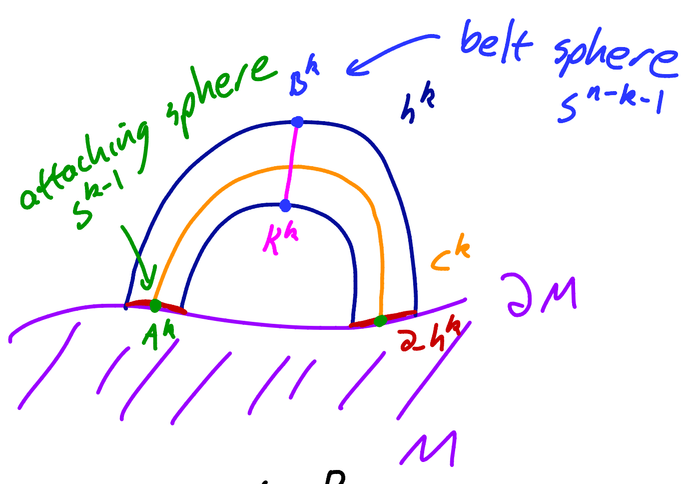
Main Facts about handle decompositions

1) all manifolds have such decompositions
   and if M compact then only finitely many handles
2) May always assume handles of index k are attached
   before handles of index $\ell > k$ and k-handles
   can be attached in any order
3) If $M' = M \cup h^k \cup h^{k+1}$ such that the attaching sphere
   of h^{k+1} intersects the belt sphere of h^k exactly
   once and transversely, then $M' \cong M$

we have already proven 1)

for 2) we need to recall

suppose h^k attached after h^ℓ for $\ell \geq k$



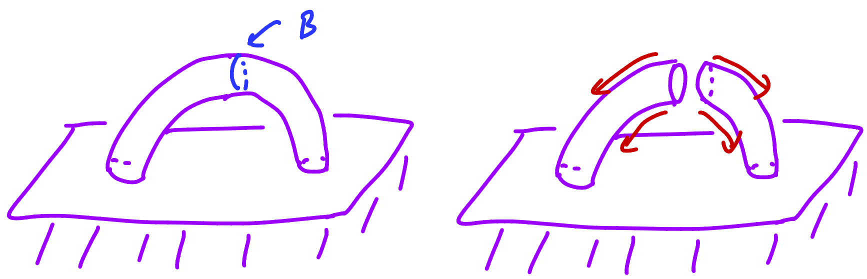
note: $A^k = S^{k-1} \subset (\partial M \cup h^\ell)^{n-1}$
$B^\ell = S^{n-\ell-1} \subset$

we can isotop A to be transverse to B

then $A \cap B$ has $\dim = (n-1) - (n-k+1) - (n-n+\ell-1)$
$= k - \ell - 1 < 0$

so A is disjoint from B

<u>exercise</u>: if $f: N \to (M \cup h^\ell)$ an embedding disjoint from B^ℓ, then f can be isotoped so $\mathrm{im}\, f$ disjoint from h^ℓ



$\therefore$ attaching region of h^k can be moved away from h^ℓ and so h^k and h^ℓ can be attached in any order

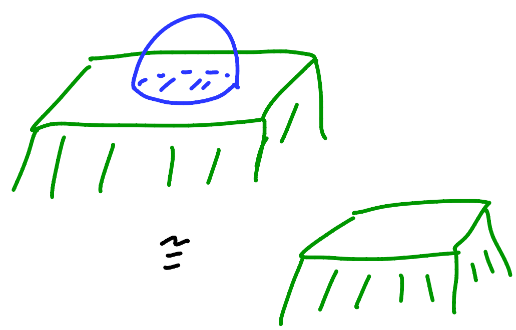
<u>Note</u>: Not true if $k > \ell$!

h^1 must come before h^2!

for 3) we need

<u>lemma</u>:

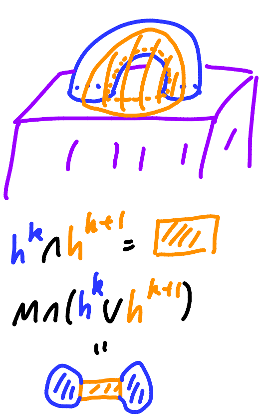
If M is an n-manifold with boundary,
A = disk $D^{n-1} \subset \partial D^n$, and
$f: A \to \partial M$ is an embedding
then $M \cup_f D^n \cong M$



<u>exercises</u>:

1) Prove this lemma

2) Under the hypothesis of 3) show you can isotop attaching regoin of h^{k+1} so $h^k \cap h^{k+1}$ is a disk
$\therefore$ by lemma $h^k \cup h^{k+1} \cong D^n$



3) Show $(h^k \cup h^{k+1} \cong D^n) \cap M$ is a disk
$\therefore$ by lemma $M' \cong M$

Corollary:

a connected cobordism W^n has a handle decomposition with

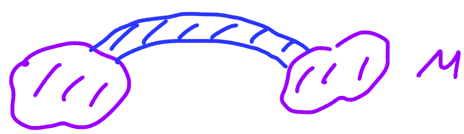
$\left.\begin{matrix}1\\0\end{matrix}\right\}$ 0-handles if $\partial_- W \begin{cases} = \emptyset \\ \neq \emptyset \end{cases}$

$\left.\begin{matrix}1\\0\end{matrix}\right\}$ n-handles if $\partial_+ W \begin{cases} = \emptyset \\ \neq \emptyset \end{cases}$

Proof: note: if $k>1$, then $\partial_- h^k$ is connected so attaching h^k must be done to a component of M

$\therefore$ $M \cup h^k$ has same number of components as M

while attaching a 0-handle adds components to M and attaching a 1-handle can either keep number of components same or reduces by 1



now assume $\partial_- W = \emptyset$, we can attach all zero handles first

if there is more than one, then W will not be connected unless $\exists$ 1-handle h^1 connecting 2 of the 0-handles h_1^0, h_2^0

note: belt sphere $= \partial h_i^0$

attaching sphere of $h^1 = S^0$

so by Fact 3 above can cancel $h^1 \cup h_2^0$

note: given a Morse function f, $-f$ is Morse too with same critical points but if one was index k for f then its $n-k$ for $-f$

similarly if you think of a handlebody "upside down" you have same handles but k handle becomes $n-k$

$\therefore$ also done with $\partial_+ W = \emptyset$

exercise: do other cases