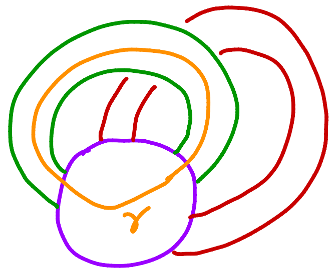
so $f_1 : F \to F$ is a diffeomorphism of F!

exercise:

1) Show f_1 is isotopic to a "Dehn twist" about γ



2) Show any diffeomorphism of an oriented surface is obtained by handle slides
(is it true for non orientable?)

E. h-cobordism theorem

the goal of this section is to prove

Th$^{\underline{m}}$ (h-cobordism theorem)

If W is an n-dimensional cobordism s.t.

1) $\pi_1(W) = \{e\} = \pi_1(\partial_\pm W)$
2) $H_*(W, \partial_- W; \mathbb{Z}) = 0$
3) $n \geq 6$

Then $W \cong \partial_- W \times [0,1]$

note: hypothesis equivalent to $\partial_\pm W \hookrightarrow W$ $\underline{\text{homotopy}}$ equivalence and $\pi_1(W) = \{e\}$

Cor:

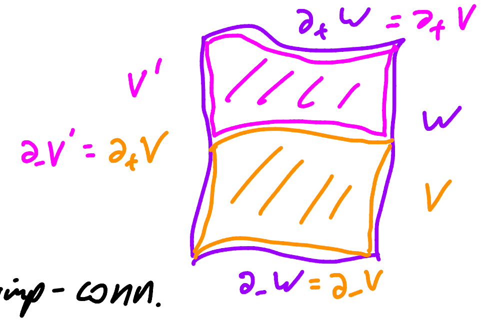
1) Suppose $V^n \subset W$ with $\partial_- V^n = \partial_- W^n$
 a) $V^n \hookrightarrow W^n$ is a homotopy equivalence
 b) $V, \partial_+ V, \partial_+ W$ are simply-connected
 c) $n \geq 6$
 $\Big\} \Rightarrow V \cong W$

2) Suppose $M^k \subset W^n$ is a submanifold with $\partial M = \emptyset$
 a) $M \hookrightarrow W$ a homotopy equivalence
 b) $M, \partial W$ simply-connected
 c) $n - k \geq 3$, $n \geq 6$
 $\Big\} \Rightarrow W \cong$ disk bundle with zero-section M.

Proof: 1) let $V' = \overline{W - V}$

$\pi_1(W) = \underset{\{e\}}{\underset{\shortparallel}{\pi_1(V)}} *_{\underset{\{e\}}{\underset{\shortparallel}{\pi_1(\partial_+ V)}}} \pi_1(V')$

$= \pi_1(V')$ so V' simp-conn.



w.l.o.g. can assume $\partial_+ V \cap \partial_+ W = \emptyset$ by adding collar to $\partial_+ W$.

$V \hookrightarrow W$ homotopy equivalence $\Rightarrow H_*(V', \partial_- V') \cong H_*(W, V) = 0$

$\therefore V' = \partial_+ V \times [0,1]$ and $V \cong W$.

2) let V = tubular nbhd of M in W

since $n-k \geq 3$, ∂V simply connected

$\therefore$ by last result $V \cong W$ ▦

Cor:

1) If W^n is contractible, $\pi_1(\partial W) = 0$, $n \geq 6$, then $W \cong D^n$

2) If W is a homotopy sphere, $n \geq 6$, then W is diffeomorphic to 2, D^n's glued together along their boundary

$\therefore W$ is homeomorphic to S^n (Poincaré Conjecture)

Proof: 1) $pt \hookrightarrow W$ satisfies hypoth of last cor

2) $\overline{W - D^n}$ is contractible with $\pi_1(\partial) = 0 \therefore \overline{W - D^n} \cong D^n$

exercise: Show W homeo. to S^n (Alexander's Trick) ▦

Remark: 1) In part 2 it is not always true that W diffeo to S^n

e.g. In dim 7 there are 28 non-diffeo. homotopy spheres more generally

n	1-3	4	5-6	7	8	9	10	11	12	13	14	15
size of "group of homotopy spheres"	1	?	1	28	2	8	6	992	1	3	2	16256

2) Proof of Poincaré in dim 5 a little harder

suppose W^5 a homotopy sphere

let $V = \overline{W - D^5}$

need to see $V \cong D^5$

note $V \times [0,1] \cong D^6$ and

$\partial(V \times [0,1]) = S^5$

$\partial V = \partial V \times \{0\}$ a smooth S^4 in S^5

Schoenflies thm (Brown, Mazur) $\Rightarrow \partial V$ bounds a D^5

in S^5 $\therefore$ $\underline{V \cong D^5}$ ✓

Algebraic Topology of Handlebodies

Recall CW-homology: X a relative CW-complex if it is obtained from Y by attaching k-cells

a k-cell (index) is $c^k = D^k$ and it is attached to a topological space A

by $A \cup D^k /_{f(p) \sim p}$, $p \in \partial D^k$

and $f : \partial D^k \to A$

can attach cells in increasing index

let $X^{(k)} = Y \cup$ all cells of index $\leq k$

let $C_k(X,Y)$ = free abelian group gen by k cells of (X,Y)

$\partial_k : C_k(X,Y) \to C_{k-1}(X,Y)$

$$c_i^k \longmapsto \sum_{c_j^{k-1}} \deg(f_{ij})\, c_j^{k-1}$$

$$\begin{tikzcd} \partial c_i^k = S^{k-1} \arrow[r, "f"] \arrow[rrr, bend right, "f_{ij}"'] & X^{(k-1)} \arrow[r] & X^{(k-1)}/X^{(k-2)} = \bigvee S^{k-1} \arrow[r] & S^{k-1} \end{tikzcd}$$

choose or 1 on each cell

now $H_k(X,Y) \cong \ker \partial_k / \operatorname{im} \partial_{k+1}$

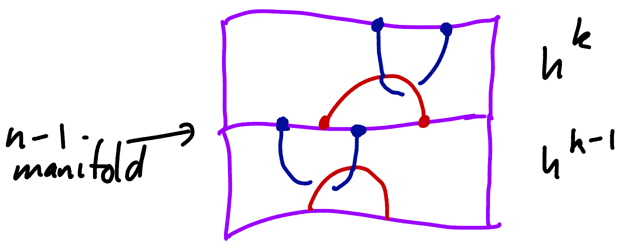
If W is a handlebody note by "collapsing" each handle to its core we get a CW-complex homotopy equivalent to W

with one k-cell for each k-handle

so $C_*(W, \partial_- W)$ = free group generated by k-handles $h^k_1, \dots, h^k_{\ell_k}$

note: attaching spheres a_i^k of h_i^k are S^{k-1}'s

belt spheres b_j^{k-1} of h_j^{k-1} are S^{n-k}



so in some level between h_j^{k-1} and h_i^k they will intersect in points

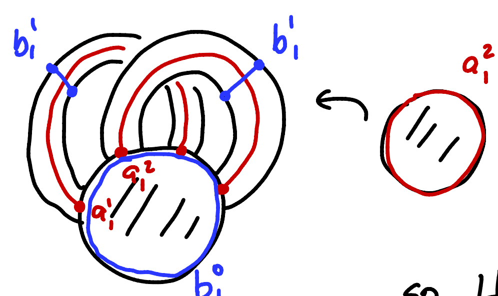
if we choose or^{n}s on all cores, we get or^{n}s on all co-cores $\therefore$ on these spheres

so we can let m_{ij}^k = intersection number.

exercise: Show the matrix (m_{ij}) represents ∂_k

examples:

1) T^2



$$C_2 \to C_1 \to C_0$$

$$\mathbb{Z} \qquad \mathbb{Z}\oplus\mathbb{Z} \qquad \mathbb{Z}$$

$1 = h_1^2 \to (0,0)$

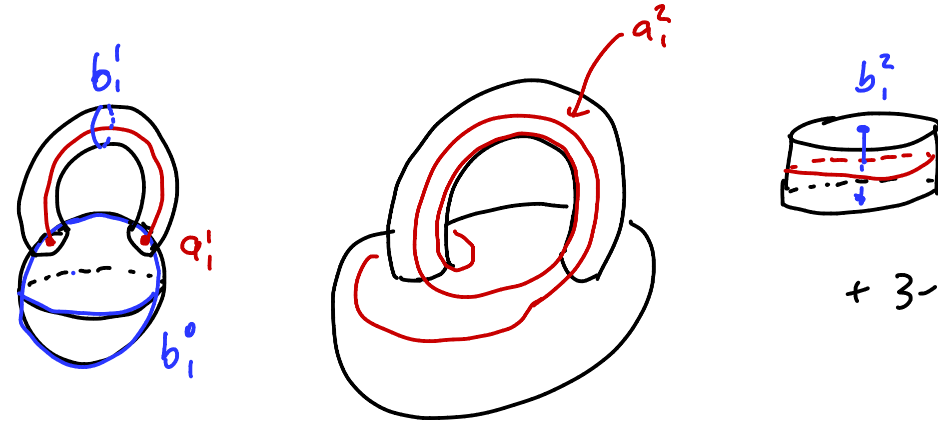
$(1,0) \to 0$

$(0,1) \to 0$

so $H_k(T^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}\oplus\mathbb{Z} & k = 1 \end{cases}$

2)

W =



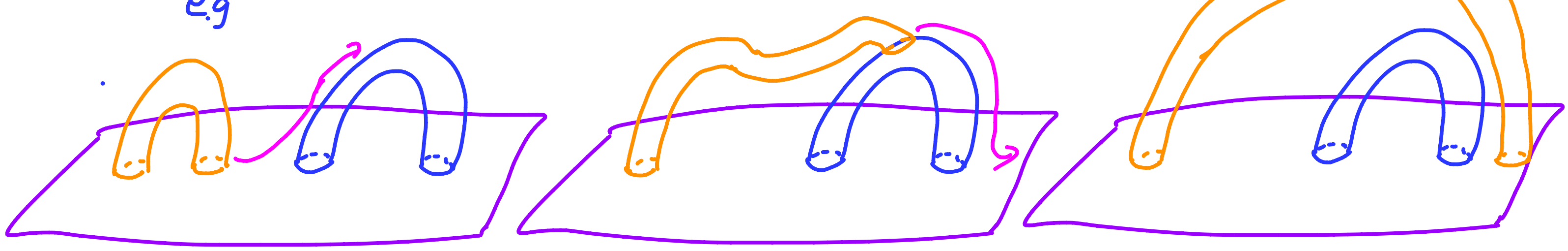
+ 3-handle

$$\begin{array}{ccccccc} C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\ 1 & \longmapsto & 0 & & 1 & \longmapsto & 0 \\ & & 1 & \longmapsto & 2 & & \end{array}$$

so $H_k(W) = \begin{cases} \mathbb{Z} & k = 0, 3 \\ \mathbb{Z}_2 & k = 1 \\ 0 & \text{other} \end{cases}$

Recall a handle slide is when we "push" a $(k-1)$-handle over another $(k-1)$-handle

e.g



now note what happens if attaching spheres of a k-handle go over these

exercise:

1) handle sliding k^{th} handle over l^{th} affects the matrix (m_{ij}) by adding k^{th} column to l^{th} column

2) Show $H_k(W, \partial_- W) \cong H^{n-k}(W, \partial_+ W)$

(recall, turning handlebody "upside down" sends k-handle to $n-k$ handle what happens to $(m_{ij}\cdot)$)

We now turn to π_1

exercise: 1) If $\partial_- W$ is connected (and no 0-handles) show that adding a

1-handle adds a generator to π_1

2-handle " " relation to π_1

k-handle $k \geq 3$, has no effect on π_1

2) If $W_k = \partial_- W \times [0,1] \cup$ all handles of index $\leq k$

then W is simply connected $\Leftrightarrow W_2$ is $\Leftrightarrow \partial_+ W_2$ is

$n \geq 5$ Hint: turn W_2 upside down

Proof of h-cobordism thm:

Given W as in thm is has the structure of a handle body

$\partial_- W \times [0,1] \cup (0\text{-h})'s \cup \ldots \cup (n\text{-h})'s$

from earlier we know we can arrange for there to be

no 0-handles and n-handles

Step 1: Get rid of all 1 an $(n-1)$-handles

we will trade 1-handles for 3-handles

Suppose we want to cancel h'

on $\partial_+ W_1$ choose a loop γ that would cancel h' if a 2-handle were attached to it

we can isotop γ so it is disjoint from the attaching regions of the 2-handles

so $\gamma \subset \partial_+ W_2$

in a ball in $\partial_+ W_2$ add a cancelling 2, 3-handle pair and let γ' be attaching sphere of 2-handle we know $\partial_+ W_2$ is simply-connected since W is

$\therefore \gamma'$ is isotopic to γ

$\therefore$ we can cancel h^1 with new 2-handle

(but this leaves an extra 3-handle)

continuing we can remove all 1-handles and

by turning W upside down we can also

remove all $(n-1)$-handles (by adding $(n-3)$ handles)

note now that $\partial_+ W_k$ simply-connected $\forall k$

Step 2: Remove 2-handles

we know $H_2(W, \partial_- W) = 0$ and since there are no 1-handles

we know $\partial_2 : C_2 \to C_1$ is 0-map

so $H_2 = C_2 / \text{im}\, \partial_3$

by using column operations (i.e. handle slides) we

can arrange $\partial_3 = \left(\begin{array}{ccc|c} \pm 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \pm 1 & \end{array} \right)$

$\therefore$ for each 2-handle h_i^2 $\exists !$ 3-handle h_i^3:

s.t. the belt sphere b_i^2 of h_i^2 intersects the

attaching sphere a_i^3 of h_i^3 algebraically once

and $b_i^2 \cdot a_j^3 = 0$ if $i \neq j$

if a_i^3 intersected b_i^2 exactly once then we could

cancel them

we can arrange this via Whitny trick

lemma:

let A^k, B^{n-k} be oriented submanifolds

of an oriented manifold W^n

suppose P, Q are transverse intersections

of opposite sign

let γ_1 be path P to Q on A

γ_2 " " Q to P on B

If $k, n-k \geq 3$, and $\gamma_1 * \gamma_2$ nullhomotopic in W

or $k=2, n\geq 5$, and $\gamma_1 * \gamma_2$ nullhomo. in $W \setminus B$

then there is an isotopy of A removing P & Q (and not adding other intersections)

recall $n-3=(n-2)-1$ sphere

so if $\partial_+ W_2 \setminus b_i^2$ is simply connected then we can use the lemma to arrange a_1^3 & b_1^2 to intersect exactly once and we are done!

s^1's

to see this note $\partial_+ W_2 \setminus (\bigcup b_i^2) \cong \partial_- W \setminus (\bigcup a_i^2)$

$$\partial_+ W_2 = [\partial_- W - (\bigcup S^1 \times D^{n-2})] \cup \bigcup (S^{n-3} \times D^2)$$

$$\partial_+ W_2 \setminus \bigcup b_1^3 = [\partial_- W - (\bigcup S^1 \times D^{n-2})] \cup (S^{n-3} \times S^1 \times [0,1))$$

$$\cong \partial_- W - (\bigcup (S^1 \times D^{n-2}))$$

$$\cong \partial_- W \setminus \bigcup a_1^2$$

but $\partial_- W$ is simply connected and the a_i's have codimension >2 so done by exercise

<u>exercise:</u> if W simply connected and $M \subset W$ has codim ≥ 3, then $W - M$ simply-connected

<u>Step 3:</u> Successively elliminate $k=3,\ldots,n-2$ handles

just like step 2 but easier (except for $k=n-2$)

since don't have to worry about codim 2 problems

So all handles are cancelled and $W \cong \partial_- W \times [0,1]$ ∎

<u>Idea for Whitney Trick:</u>

given that $\gamma_1 * \gamma_2$ is nullhomotopic $\exists$ a disk $f: D^2 \to W$ with boundary $\gamma_1 * \gamma_2$

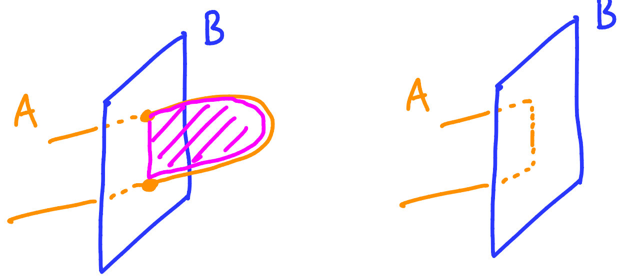
since $n \geq 5$, f can be assumed to be an embedding

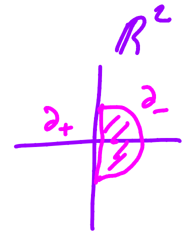
(embeddings are dense in $C^\infty(M^n, X^m)$ if $m > 2n$)

moreover, if codim $A, B > 2$, then int f disjoint from $A \cup B$

if codim $B = 2$, need to assume disk is disjoint from B

now one use D^2 to guide an isotopy of A





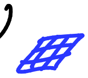
let $D^2 = \{(x,y) \mid x^2+y^2 \leq 1,\ x \geq 0\}$

can find nbhd U of $D^2 \times \{0\}$ in $D^2 \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1}$

and embedding $\phi : U \to W$ st. $\phi|_{D^2 \times \{0\}} = f$

and $\phi^{-1}(A) = U \cap (\text{''} \quad \text{''} \quad \text{''} \times \{0\})$

$\phi^{-1}(B) = U \cap (\partial_- D^2 \times \{0\} \times \mathbb{R}^{n-k-1})$

now one can write an explicit isotopy in U 

F. Heegaard Splittings of 3-manifolds

let $H_g = (0\text{-handle}) \cup g\,(1\text{-handles})$ st. H_g is oriented

this is called a genus g handlebody

notice that since any embedding of $S^0 \hookrightarrow S^2$ is isotopic to any other

← attaching sphere of 1-handle

and $\exists$ unique way to "frame" 1-handle to get oriented mfd

we see H_g only depends on g