to prove this (from our perspective) we need something new.

G. Cerf and Smale theory

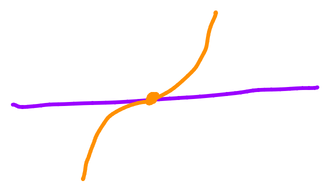
first Cerf theory addresses how to get between 2 Morse functions

recall a critical point p of $f: M \to \mathbb{R}$ is non-degenerate if $df: M \to T^*M$ is $\pitchfork$ zero section at p

if df not $\pitchfork$ at p what's the next closest thing?

"order 1 tangency, in one direction"

eg



$x \mapsto x^3$

we say p is <u>embryonic</u> if $\ker \mathrm{Hess}_p f$ is 1 dimensional and, in local coords, 3^{rd} derivative of f in direction of $\ker \mathrm{Hess}_p f$ is non-zero

we call functions f with non-degen. and embryonic critical points <u>generalized Morse functions</u>

i.e. $\operatorname{im} df_p \cap$ zero section $=$ line and in that direction "$\frac{\partial^3 f}{\partial x^3} \neq 0$"

similar to Morse lemma we have

lemma:

If p is an embryonic critical point of f, then $\exists$ local coords about p in which f takes the form

$$f(x_1, \dots, x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + x_n^3$$

k is index of p

Similar to Th^m about the existence of Morse functions we have

Th^m (Cerf):

If $f_0, f_1 : M \to \mathbb{R}$ are Morse functions, then for generic paths $f_t : M \to \mathbb{R}$ connecting f_0, f_1 we can assume $\exists$ a finite numbere of $t_1 < \dots < t_k$ such that for $t \neq t_i$, f_t is Morse with distinct critical values and for $t = t_1$ either

1) f_{t_1} critical values distinct, one critical value is birth/death and rest non-degenerate

2) f_{t_1} Morse, but exactly 2 critical points have the same value (rest distinct)

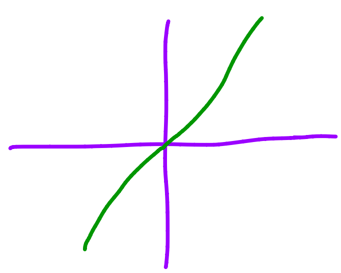
p is a birth/death critical point, if it is embryonic for f_{t_0} and $(0, p)$ is a non-degenerate critical point of $(t, p) \mapsto f_t(p)$

lemma: If $f_t : M \to \mathbb{R}$, $t \in (-\varepsilon, \varepsilon)$ is a path of functions such that for $t \neq 0$, f_t is Morse and for $t = 0$, p is an embryonic critical point of f_0, then there are coordinates about p in M and about $f(p)$ in $\mathbb{R}$, such that f_t takes the form

$$f(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + x_n^3 \pm t x_n$$

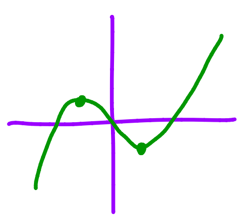
note: in $+t x_n$ case then

for $t > 0$, $x_n(x_n^2 + \varepsilon)$ has no critical points



so f has no critical points near p

for $t < 0$, $x_n(x_n^2 - \varepsilon)$ has 2 critical points



so f has 2 critical points near p of index k and $k+1$

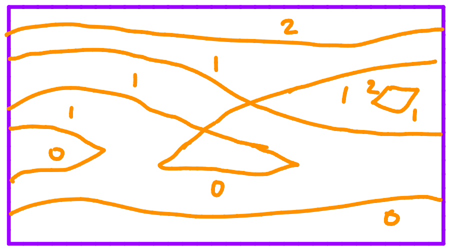
one may easily check that these critical points give canceling handles!

call such a critical point a death point and when $t < 0$ call it a birth point it is useful to look at the Cerf graphic

that is for $f_t : M \to [a,b]$, $t \in [0,1]$

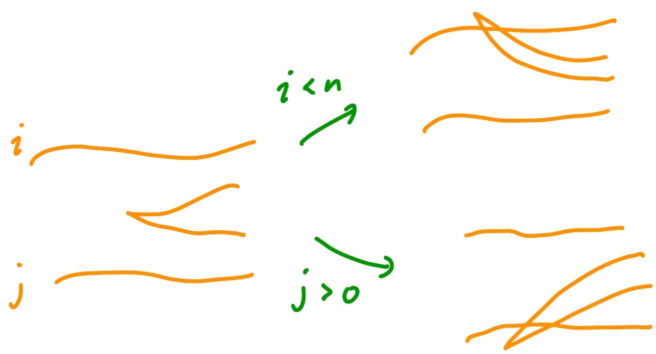
plot the critical values of f_t and lable w/ index

from above this could look like



here are 2 simple manipulations of the graphic

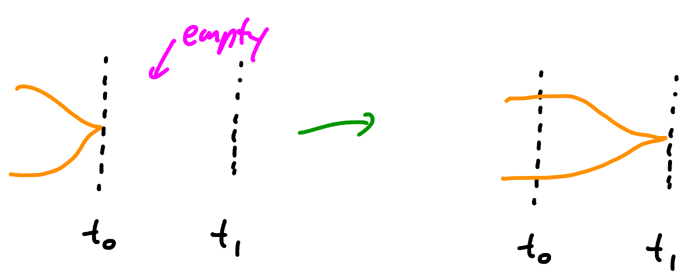
Beak move



(and reflection abt vertical line)

Independence Principle

Beak Isotopy



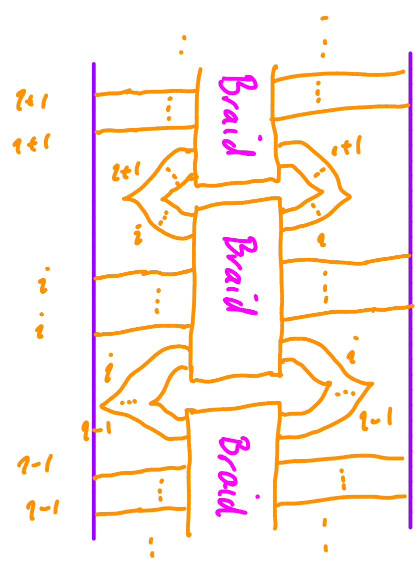
call a Morse function ordered if critical values of index i critical points are lower than all index $i+1$ critical points

we will see later (or you can think about this in terms of handle decompositions) that any M has an ordered Morse function (even more than that!)

exercise:

1) show if f_0, f_1 are ordered Morse functions then $\exists$ a path f_t of function s.t. f_t is Morse except for $0 < t_1 < \ldots < t_k < \frac{1}{3} < \frac{2}{3} < t_{k+1} < \ldots < t_{k+l} < 1$ s.t. all t_i, $i \leq k$ are births and all t_i, $i > k$ are deaths

2) Moreover, show can assume f_t, $t \neq t_i$ are ordered so Cerf graphic is



Recall to get a handlebody decomposition of W from a Morse function f we used the gradient of f w.r.t. some metric, in general we could use "gradient-like" vector fields

a vector field v is gradient-like for f if

1) $df(v) < 0$ away from crit pts of f

2) Hess $df(v)$ is negative definite at critical pts of f

exercise: 1) critical points of f and v agree

2) critical points are "hyperbolic" eigenvals of linearization have non-zero real parts

3) fixing f the space of gradient-like vector fields for f is convex (hence contractible)

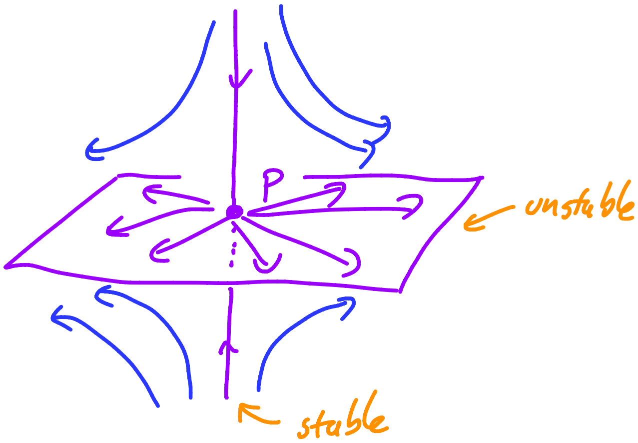
let $\phi_t : W \to W$ be the flow of v

The stable/unstable manifold Th$^{\underline{m}}$ says for each non-degenerate critical point p of index k of v the set of points

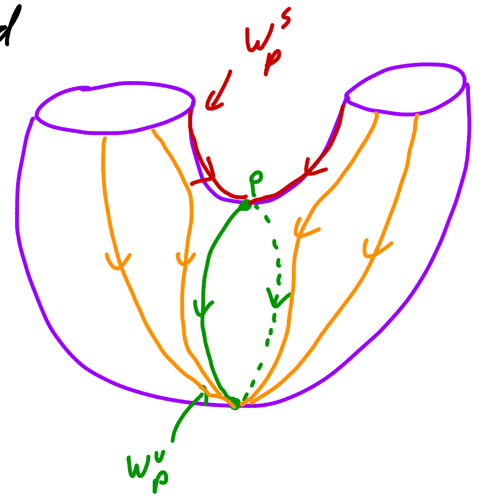
$$W_p^u = \{x \in W : \lim_{t \to -\infty} \phi_t(x) = p\}$$ unstable mfd of p

$$W_p^s = \{x \in W : \lim_{t \to \infty} \phi_t(x) = p\}$$ stable mfd of p

are manifolds diffeomorphic to $\mathbb{R}^k, \mathbb{R}^{n-k}$ respectively



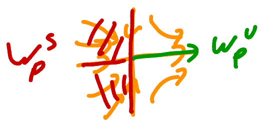
in a mfd



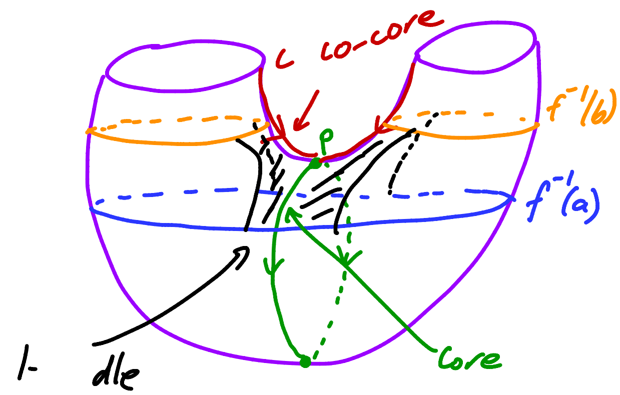
so $W_p^u \cap$ super level set = core of corresponding handle

$W_p^s \cap$ sublevel set = co-core

(for embryonic W_p^u, W_p^s mfds w/ ∂



)



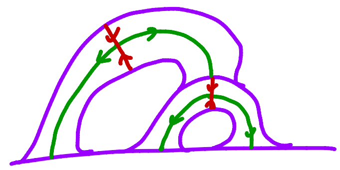
Th$^{\underline{m}}$ (Smale):

1) for any Morse function $f: W \to \mathbb{R}$, $\exists$ a gradient-like vector field such that the stable and unstable manifolds are transverse

2) for a family of generalized Morse functions f_t, $\exists$ a family of gradient-like vector fields v_t s.t. off of a finite set $t_1 < \ldots < t_k$ the stable and unstable manifolds are transverse, and at t_i, f_{t_i} is Morse and there is one pair of index i critical points s.t. stable mfd of one intersects unstable mfd of other (and "transversely")

note: 1) W_p^u, W_q^s being transverse, means $W_p^u \cap W_q^s = \emptyset$ if index $p \leq$ index q (note if $\cap$ then, dim of $\cap \geq 1$) since invt under flow.

2) non-generic moments in part 2) are handle slides



note the Th$^{\underline{m}}$s of Cerf and Smale says

Cor:

Any 2 ordered Morse functions and gradient-like vector fields $(f_0, v_0), (f_1, v_1)$ with generic stable/unstable manifolds are connected by a path (f_t, v_t) with isolated birth/deaths and handle slides

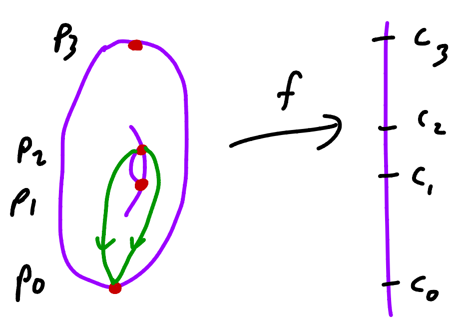
In particular, any 2 handle decompositions are related by a sequence of adding/removing cancelling pairs and handle slides

Previously we saw we could do a lot with such moves, but now we know that this is a complete set of moves!

lemma:

If the unstabele manifold of an index k critical point p intersects the regular level set $f^{-1}(a)$ in an S^{k-1} then one may alter f (near W^u_p) so that $f(p)$ is $a+\varepsilon$ for any $\varepsilon>0$ (keeping same v)

"push down critical pts"



can arrange $f(p_3)$ arbit close to c_2

and $f(p_2), f(p_1)$ arbit close to c_0

and $f(p_0)$ any $\# < c_0$

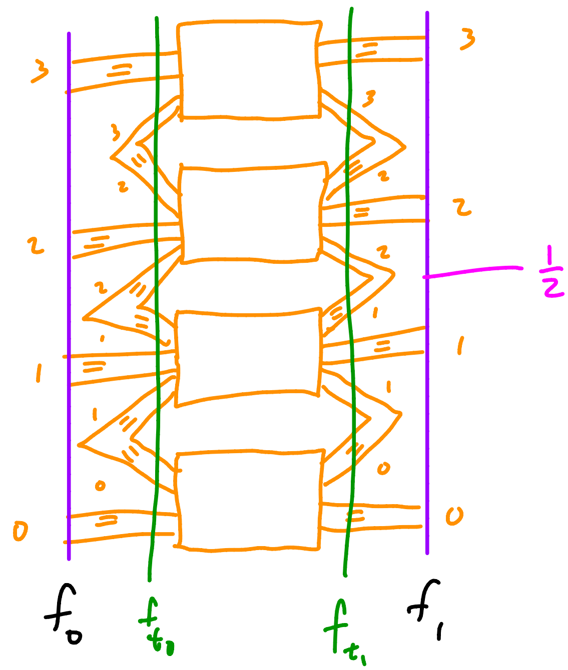
there is an analogous "push up" lemma

note we can now prove the Reidemeister-Singer theorem

Proof: given 2 Heegaard surfaces Σ_0, Σ_1 for M, there are associated Morse function $f_i : M \to [0,1]$ s.t. $f_i^{-1}(1/2) = \Sigma_i$.

let f_t be a path of functions between f_0, f_1

we can arrange the Cerf graphic for f_t to be



note that $f_{t_i}'^{-1}(1/2) = \Sigma_i'$ is a stabilization of Σ_i

note f_t for $t \in [t_0, t_1]$ are Morse and $1/2$ a regular value

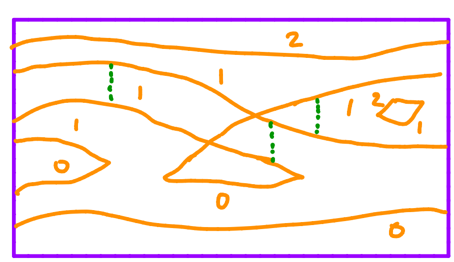
so we get $\Sigma_t' = f_t^{-1}(1/2)$

exercise: show Σ_t is an isotopy from Σ_{t_0}' to Σ_{t_1}'

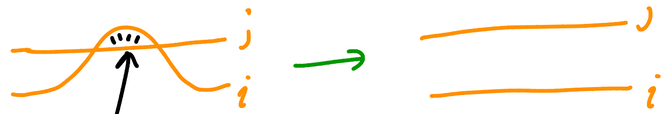
$\therefore$ done!

We can prove something stronger, for this we note

we can decorate a Cerf picture with moments of handle slides e.g.



Improved Independence Principle

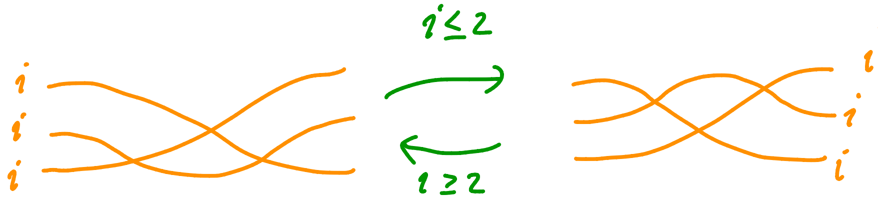


can alway do as long as contains no $\cap$ of stable and unstable manifolds

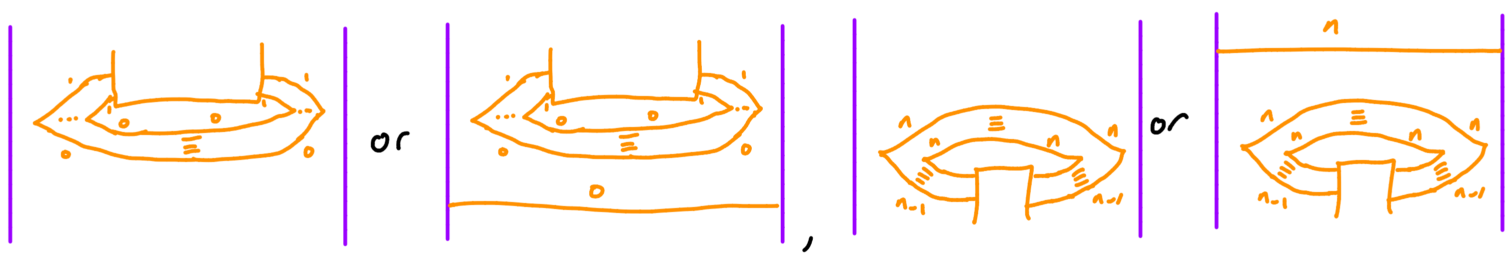
note: an index 0 critical point has $W^u \neq \emptyset$
(" " n " " $W^s \neq \emptyset$)

also have

triangle move

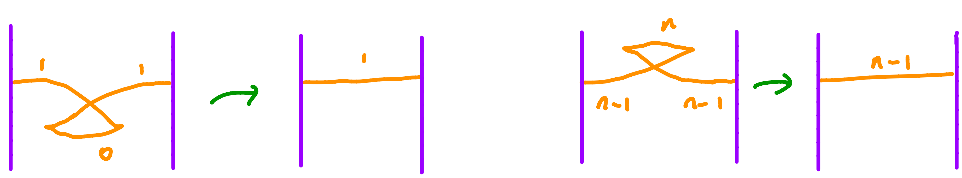


exercise: show if f_0, f_1 have 0 or 1 index 0 or n crit pt you an alway improve our normalized Cerf graphic too:



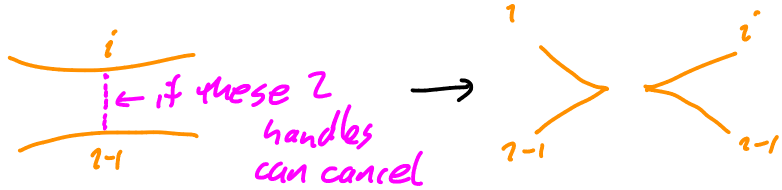
finally we need

Swallowtail move



more general move but need extra conditions

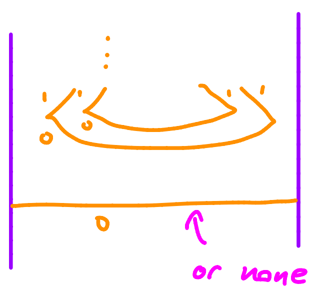
Cancellation move

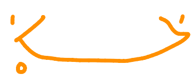


lemma:

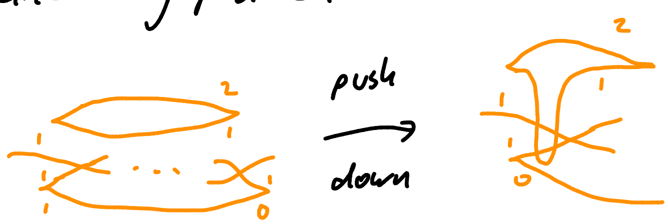
> If f_0, f_1 are Morse functions both having 0 or 1 critical points of index 0 (or n), then there is a path f_t of (generalized) Morse functions s.t. the number of index 0 (or n) critical points is constant in t

Proof: we know we can arrange

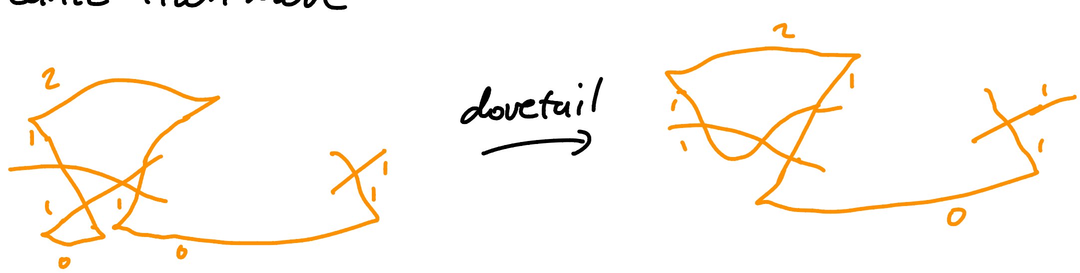


Consider upper most  looks like 

now add a 1,2-cancelling pair so the added 1-hande is in cancelling position for the 0-handle



do the cancellation move



now many triangle moves get us to

so we have elliminated one 

inductively get rid of all of them

Thm:

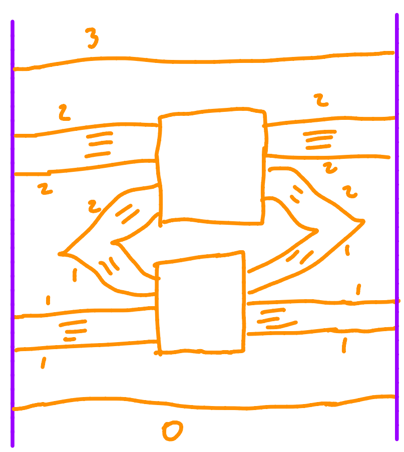
any 2 Heegaard diagrams for M are related by

a) birth/death of 1,2 handle pairs

b) handle slides

Proof: given 2 Heegaard diagrams we get 2 corresponding Morse functions f_0, f_1 each having a single index 0 and 3 critical pt

from above we have a family f_t with Cerf graphic



from which the theorem follows