PLANAR OPEN BOOKS AND SYMPLECTIC FILLINGS

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To my parents

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ABSTRACT

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A construction is given that shows an interesting family of Stein fillable contact 3-manifolds are supported by planar open books. Knowing that a Stein fillable contact 3-manifold is supported by a planar open book might enable one to determine the diffeomorphism types of fillings. We demonstrate this technique by showing that the small Seifert fibred space $M(-3; -2, -2, -2)$ has a unique Stein filling.

Contents

List of Figures

Chapter 1

Introduction

The correspondence between contact structures and open books, developed by Giroux in 2000, created a new burst of work in contact topology. Recently, Abbas et. al. [1] proved the Weinstein conjecture for contact 3-manifolds that are supported by planar open books, i.e. open books whose page is a punctured 2-sphere. That this does not solve the Weinstein conjecture in generality was shown by Etnyre [8] who provided the first obstructions for fillable (and hence tight) contact structures to be supported by planar open books. He did so by showing that such fillings can be compactified to a blow up of a ruled symplectic manifold, which proves that, among other things, the intersection form of a filling must have $b_2^+ = 0$. Etnyre's proof uses a result by Eliashberg [6], a construction to cap off the boundary of an open book by 2-handles in a symplectic way, thus giving rise to a symplectic cobordism of the 3-manifold to a surface bundle over the circle. Such surface bundles can then be capped off to yield a closed symplectic manifold. This work first constructs planar open books for a family of Stein fillable contact manifolds, among them Lens spaces and small Seifert fibred spaces with $e_0 \leq -3$.

The problem to determine the diffeomorphism types of fillings for a contact 3-manifolds is interesting and intriguing. It was first shown by Eliashberg [4] that any Stein filling of the tight contact 3-sphere is diffeomorphic to the 4-ball. McDuff [24] showed that for contact structures on Lens spaces $L(p, 1)$ that are quotients of a cyclic action on (the unique) tight contact structure on S^3 there is a unique (up to blow up) diffeomorphism type of fillings if $p \neq 4$ and there are two in case $p = 4$. Her argument showed that such fillings can be compactified to a ruled symplectic manifold and identified the complement of a filling to be a neighborhood of a symplectic sphere with self intersection $p > 0$. Such configurations of symplectic spheres are unique up to isotopy and this proves her result about the filling itself. Recently, Hind [17] has shown that these fillings are unique up to Stein homotopy. This result was proved for $L(2, 1)$ three years earlier [16] using similar techniques.

Lisca [22] generalized McDuff's result, using a similar line of argument, to contact structures on all Lens spaces that arise as quotients of the tight contact structures on S^3. To do this, Lisca used a glueing result by McCarthy and Wolfson to construct compactifications of symplectic fillings. Also Ohta and Ono [29] study diffeomorphism types of fillings for contact structures from Milnor fibers in a similar way. Most of the examples above used a compactification to a ruled surface.

Using ad-hoc methods to embedd Stein fillings of T^3 into homotopy $K3$ surfaces, Stipsicz showed that a Stein filling of the 3-torus T^3 with the contact structure $\xi = \ker(\cos(2\pi z)dx + \sin(2\pi z)dy)$ is homeomorphic to $T^2 \times D^2$.

We use McDuff's strategy for solving this problem, but use Etnyre's construction to provide compactifications to a ruled surface. Doing this carefully allows to determine the complement of a filling and classify fillings up to diffeomorphism. This possibly allows to provide a diffeomorphism classification of fillings for tight contact 3-manifolds supported by planar open books. We illustrate this technique in case of the small Seifert fibred space $M(-3; -2, -2, -2)$.

Chapter 2

Background

In this chapter we describe some background results about contact and symplectic topology that are needed throughout this thesis.

2.1 Contact structures and open books

For a good introduction to open books and contact structures, the reader is advised to [7, 30]. To learn how to see open books from a Kirby diagram see [11]. We assume the reader is familiar with Kirby Calculus, see [15, 30]. Unless stated otherwise, suppose that 3-manifolds compact and oriented and that contact structures are positive. We orient symplectic manifolds using the symplectic structure, i.e. such that $\omega \wedge \omega > 0$.

2.1.1 Contact manifolds and their fillings

If (X, ω) is a symplectic 4-manifold and v is a Liouville vector field defined in a neighborhood of ∂X, then v defines a contact form $\alpha = i_v\omega|_{\partial X}$ on ∂X. In case $Y = \partial X$ is connected and v points out of (into) X along ∂X, we call (X, ω) a *convex* (*concave*) *filling* of the contact manifold $(Y = \partial X, \xi = \ker(\alpha))$. Assume we have a convex filling (X, ω) for a contact manifold (Y, ξ) and there is furthermore an almost complex structure J on X and a strictly plurisubharmonic function $\varphi : X \to \mathbb{R}$ such that $\omega = -dJ^*d\varphi$ and Y is a regular level set of φ, then we call X a *Stein filling*. We refer to [6] for the current state of art for different notions of fillability and their relations among eachother.

Example 2.1.1. Consider the 3-sphere $S^3 \subset \mathbb{C}^2$. In polar coordinates $(r_1, \theta_1, r_2, \theta_2)$ the standard symplectic structure on $\mathbb{C}^2$ is given by $\omega = r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2$. The vector field $v = \frac{1}{2}(r_1^2\partial_{r_1} + r_2^2\partial_{r_2})$ is easily verified to be a Liouville vector field, i.e. that $L_v\omega = \omega$, and is certainly defined in a neighborhood of $S^3 = \{r_1^2 + r_2^2 = 1\}$. The contact form $\alpha = i_v\omega|_{S^3} = \frac{1}{2}(r_1^2 d\theta_1 + r_2^2 d\theta_2)$ gives rise to the standard tight contact structure on S^3. Notice that the function $\varphi : \mathbb{C}^2 \to \mathbb{R}$ given by $\phi(r_1, \theta_1, r_2, \theta_2) = r_1^2 + r_2^2$ is a strictly plurisubharmonic function for the standard complex structure on $\mathbb{C}^2$ and S^3 is a regular level set. Thus the 4-ball $D^4 = \{r_1^2 + r_2^2 \leq 1\} \subset \mathbb{C}^2$ is indeed a Stein filling.

Suppose (X, ω) is a symplectic manifold with boundary $\partial X = (-Y_1) \sqcup Y_2$ for two 3-manifolds Y_1 and Y_2. Moreover assume there exists a Liouville vector field v

defined in a neighborhood of ∂X pointing into X along Y_1 and pointing out of X along Y_2. Then (X, ω) is called a *symplectic cobordism* from (Y_1, ξ_1) to (Y_2, ξ_2).

Example 2.1.2. Given a contact manifold $(Y, \xi = \ker(\alpha))$ one can construct a symplectic manifold $(\mathbb{R}\times Y, \omega = d(t\alpha))$, where t denotes the $\mathbb{R}$-coordinate. Then $v = t\partial_t$ is a Liouville vector field for ω. For two positive functions $f_1,\ f_2 : Y \to \mathbb{R}$ with $0 < f_1 < f_2$, we find a symplectic cobordism $(X, \omega) = \{(t, p) \in \mathbb{R} \times Y \ : \ f_1(p) \leq t \leq f_2(p)\}$ from $(Y, f_1\alpha)$ to $(Y, f_2\alpha)$.

Self-cobordisms as in Example 2.1.2 are important to establish certain properties for the contact form of a given contact manifold (Y, ξ). This allows for symplectic cobordisms to be glued together.

2.1.2 Open books

Before turning to a discussion of open books, we recall some facts about diffeomorphisms of surfaces. For a compact orientable surface F with boundary we define the *mapping class group* $M(F)$ of F as the group of orientation preserving diffeomorphisms of F fixing the boundary pointwise, up to isotopies of such diffeomorphisms. We will generally not make a distinction between a diffeomorphism and its class in the mapping class group. If $F' \subset F$ is a subsurface, then there is a natural inclusion $M(F') \subset M(F)$ by extending diffeomorphisms of F' by the identity to F. Furthermore recall that

Theorem 2.1.3 (Dehn [2], Lickorish [21]). *The mapping class group is generated*

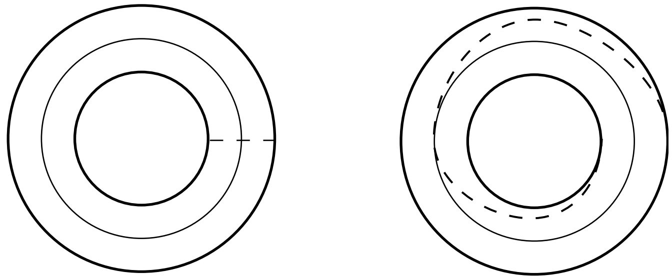

Figure 2.1: Positive Dehn twist about c

by Dehn twists.

For a simple closed curve $c \subset F$, identify a neighborhood of c with $S^1 \times [0,1]$ with $c = S^1 \times \frac{1}{2}$. Then a *positive Dehn twist about* c is given by $t_c(\theta, t) = (\theta + 2\pi t, t)$, shown in Figure 2.1. Its inverse, a negative Dehn twist, will be denoted by t_c^{-1}. Notice that a Dehn twist only depends on the orientation of F, but not on the orientation of c.

Definition 2.1.4. An *open book* (B, π) for a 3-manifold Y is a given by an oriented link $B \subset Y$ together with a fibration $\pi : Y \setminus B \to S^1$ of its complement with fiber $\pi^{-1}(\theta) = \text{int}(F_\theta)$ the interior of a compact surface $F_\theta \subset M$ with oriented boundary $\partial F_\theta = L$. We call F_θ a *page* of the open book and B the *binding*.

Notice that around each component of B one finds a neighborhood $N = S^1 \times D^2$ with cylindrical coordinates (r, ϑ, z), such that $F_\theta \cap N = \{\vartheta = \theta\}$. Furthermore any vector field v transverse to the pages and meridional in a neighborhood of B gives rise to a return map $\phi_v : F_0 \to F_0$ where F_0 is a page of the open book. ϕ_v is called the *monodromy* of the open book.

Definition 2.1.5. An *abstract open book* (F, ϕ) for a 3-manifold Y is given by a diffeomorphism $\phi : F \to F$, called the *monodromy* of a compact oriented surface F with $\phi|_{\text{nbhd}(\partial F)} = Id$. Then

$$Y = F \times [0,1]/_{(\phi(p),0)\sim(p,1)} \cup_\psi \left(\bigsqcup_{|\partial F|} S^1 \times D^2 \right),$$

where ψ is minus the identity identifying the torus boundaries, when equipped with the induced product structures. Notice that ψ in the definition above is unique up to isotopy. Again, we call a surface $F \times \{p\}$ a *page* of the open book, for $p \in [0,1]/\sim$, and ∂F the *binding*.

Two abstract open books (F_1, ϕ_1) and (F_2, ϕ_2) are called *equivalent* if there is a diffeomorphism $f : F_1 \to F_2$ such that $f \circ \phi_1 = \phi_2 \circ f$. The following Lemma collects basic facts about (abstract) open books, c.f. [7, Lemma 2.4].

Lemma 2.1.6. *(a) An open book (B, π) gives rise to an abstract open book (F, ϕ).*

(b) An abstract open book (F, ϕ) determines Y and an open book up to diffeomorphism.

(c) Equivalent open books give diffeomorphic manifolds.

It is important to keep the following sublety in mind: while open books are given up to isotopy, abstract open books are determined up to diffeomorphism. This will become important, since we will have the case where the same abstract open book is compatible with contactomorphic contact structures, but these contact structures are not isotopic.

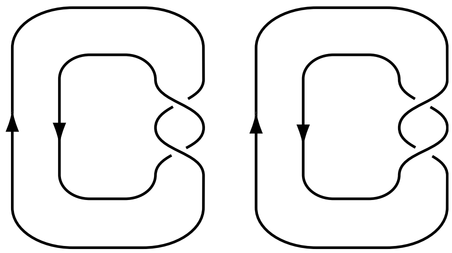

Figure 2.2: Positive and negative Hopf links.

Example 2.1.7. Consider $S^3 \subset \mathbb{C}^2$. Define $D = \{(r_1, \theta_1, r_2, \theta_2) \in S^3 \ : \ r_1 = 0\}$, $H^+ = \{(r_1, \theta_1, r_2, \theta_2) \in S^3 \ : \ r_1 r_2 = 0\}$ and similarly $H^- = \{(r_1, \theta_1, r_2, \theta_2) \in S^3 \ : \ r_1 r_2 = 0\}$. Notice that D is an unknot, and $H^\pm$ is called the *positive*, respectively *negative Hopf link*, with orientations shown in Figure 2.2. There are fibrations

$$\pi_0 : S^3 \setminus D \to S^1 \ : \ (r_1, \theta_1, r_2, \theta_2) \mapsto \theta_1$$

$$\pi_+ : S^3 \setminus H^+ \to S^1 \ : \ (r_1, \theta_1, r_2, \theta_2) \mapsto \theta_1 + \theta_2$$

$$\pi_- : S^3 \setminus H^- \to S^1 \ : \ (r_1, \theta_1, r_2, \theta_2) \mapsto \theta_1 - \theta_2.$$

All of these fibrations give rise to an open book of S^3 with page a disk in the case of π_0 and an annulus in the case of $\pi_\pm$. The monodromy of π_0 is the identity. The monodromy of the open book $(H^\pm, \pi_\pm)$ is a positive Dehn twist t_c about the core curve c of the annulus, respectively a negative Dehn twist about this core.

2.1.3 Open books supporting contact structures

Definition 2.1.8. A contact structure ξ on Y is called *compatible* with an open book (F, ϕ) if $d\alpha$ gives a volume form on the pages F and $\alpha|_{\partial F} > 0$. Vice versa, we say an open book decomposition of Y *supports* ξ.

Example 2.1.9. Consider $S^3 \subset \mathbb{C}^2$ as in Example 2.1.7 with the standard tight contact structure ξ as in Example 2.1.1. Then one can show that π_0 and π_+ are compatible with ξ, while π_- is not.

Already Thurston and Winkelnkemper [31] showed that an open book decomposition of an orientable 3-manifold gives rise to a compatible contact structure. Torisu [32] & Giroux [13] observed that this contact structure is unique up to isotopy. Giroux also proved the converse, which makes this relationship most useful.

Theorem 2.1.10. *Every open book decomposition of a* 3*-manifold supports a contact structure, unique up to isotopy. Any contact structure is supported by an open book.*

Stein fillable contact 3-manifolds are characterized by the monodromy of a supporting open book. Namely:

Theorem 2.1.11. *A contact structure is Stein fillable if and only if there is a compatible open book whose monodromy can be expressed as a product of positive Dehn twists.*

Refining results by Weinstein [33] and Eliashberg [5] about symplectic cobordisms, one obtains the following results; see [9, 10].

Theorem 2.1.12. *Suppose* (Y, ξ) *is a contact manifold compatible with an open book* (B, π) *with page* F'*. Assume the surface* F *is obtained from* F' *by attaching* *a 1-handle at points p, $q \in \partial F'$. Then there exists a symplectic cobordism (X, ω) which is diffeomorphic to $Y \times [0, 1]$ with a 4-dimensional 1-handle attached to $Y \times 1$ at p, q.*

Recall that a 1-dimensional submanifold in (Y, ξ) that is tangent to ξ is called *Legendrian*. Given a simple closed curve in a page of an open book decomposition of Y supporting ξ, we can employ the Legendrian realization principle [19, Theorem 3.7], to isotope this curve to be Legendrian, given that it is homologically nontrivial in $F \setminus \partial F$. Notice that a knot contained in a page obtains a framing from a vector field tangent to the page and transverse to the knot. This framing is called the *page framing*, $p(c)$.

Theorem 2.1.13. *Suppose (Y, ξ) is a contact manifold compatible with an open book (B, π) with page F and monodromy ϕ. Set $\phi' = \phi \circ t_c$, where c is a homologically nontrivial curve in $F \setminus \partial F$ and t_c denotes a positive Dehn twist along c. Denote by (Y', ξ') the contact manifold supported by the open book (F, ϕ'). Then there is a symplectic cobordism (X, ω), from (Y, ξ) to (Y', ξ'), diffeomorphic to $Y \times [0, 1]$ with a 4-dimensional 2-handle attached to c with framing $p(c) - 1$.*

Given a contact 3-manifold (Y, ξ) compatible with an open book decomposition (F, ϕ), consider the open book (F', ϕ') obtained from (F, ϕ) as follows. First attaching a 1-handle as in Theorem 2.1.12 yields F'. Second, choose a curve c contained in a page that runs over this 1-handle once and set $\phi' = \phi \circ t_c$. According to Theorem 2.1.13, performing this positive Dehn twist along c can be achieved by attaching a

2-handle. Denote the contact 3-manifold obtained from (F', ξ') by (Y', ξ'). Since the curve chosen to attach the 2-handle runs over the 1-handle once, this handle pair is canceling, and so Y is diffeomorphic to Y'. Furthermore, by a theorem of Giroux [13], the contact structures ξ and ξ' are isotopic. We say (F', ϕ') is obtained from (F, ϕ) by *stabilization*.

Example 2.1.14. We can visualize the open books from Example 2.1.7 in an abstract way. The first one, π_0, is obtained as follows. We find an open book on $\mathbb{R}^3$ with cylindrical coordinates (r, ϑ, z) where the z-axis is the binding and the pages are given by $\vartheta = const$. Adding a point at infinity, one obtains the open book π_0 on S^3. To obtain a 4-dimensional picture of π_+, we stabilize the open book π_0 once. To do this, think of the 4-ball as the product of a 2-dimensional 0-handle and a 2-disk, and of a 4-dimensional 1-handle as the product of a 2-dimensional 1-handle and a 2$-$disk. Thus by attaching a 1-handle to a 0-handle we see a disks worth of annuli. Second, pick one of these annuli and attach a 2-handle to its core, with framing (-1) with respect to the page. Eventually we can exhibit a 4-dimensional manifold whose boundary carries an open book, namely S^3 with its open book from the positive Hopf link, this is shown in Figure 2.3.

2.1.4 Stabilizing Legendrian knots via open books

We will also make use of a relation between the stabilization of open books and stabilization of Legendrian knots contained in a page, see [7]. For a Legendrian knot

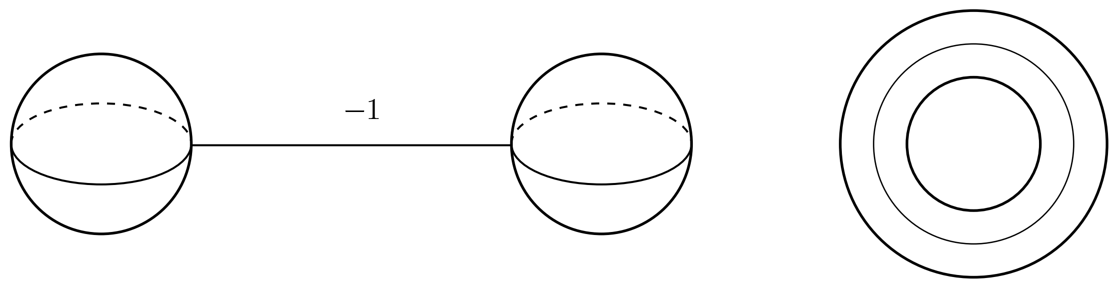

Figure 2.3: A 4-dimensional and an abstract picture of the open book (H^+, π_+). (In the abstract picture, the monodromy is a positive Dehn twist about the core of the annulus.)

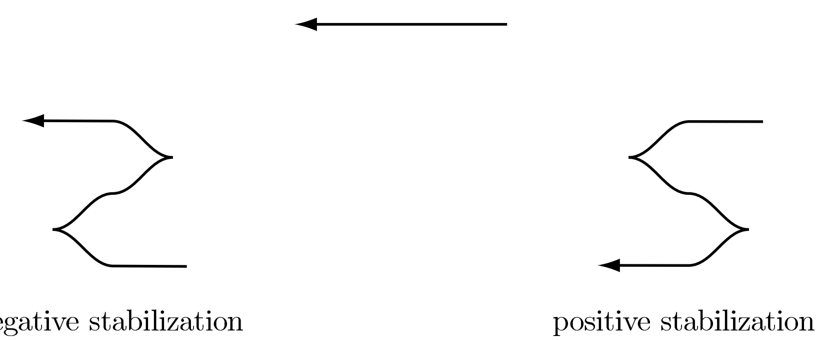

Figure 2.4: Positive and negative stabilization of a Legendrian knot in the front projection.

in $(\mathbb{R}^3, \xi_{st} = \ker(dz - ydx))$ one can define *stabilization*, using front projections, by adding zig-zag's. There are two versions, both decreasing the Thurston-Bennequin number by one, but *positive* (*negative*) stabilization increases (decreases) the rotation number by one, see Figure 2.4. Given a Legendrian knot on a page of an open book, we can stabilize it by first stabilizing the open book and then Legendrian realizing a curve running over the 1-handle once, see Figure 2.5.

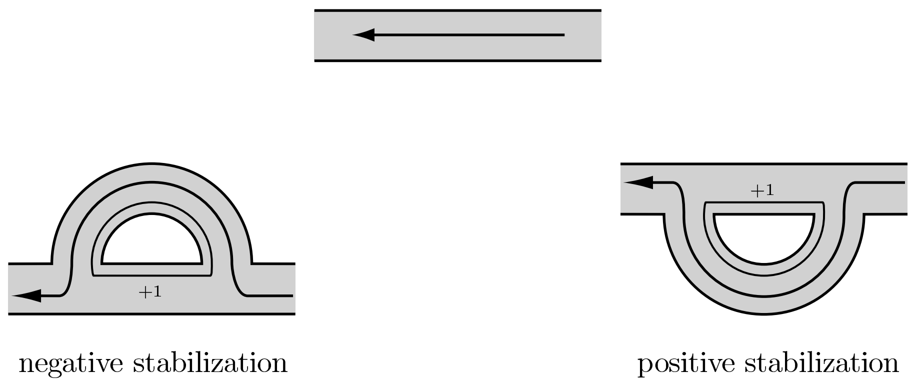

Figure2.5: Positive and negative stabilization of a Legendrian knot contained in a page of an open book.

2.1.5 E-handles and capping off symplectic fillings

The last paragraph in this section is devoted to explain the attachment of *Eliashberg-handles* (which we will call *E-handles* for short). Notice that, topologically, 0-surgery along the binding of an open book decomposition of a 3-manifold yields a surface bundle over the circle. Eliashberg [6] showed that this can be done in a symplectic way. Eventually one can cap off this surface bundle to show that any (weakly) fillable contact 3-manifold embedds into a closed symplectic manifold. Although the importance of this result cannot be overemphasized, for the purpose of this thesis we content ourselves to recall the construction of these handles.

Pick a constant $k > 0$ and consider a function $g : [0, 1] \to \mathbb{R}$ that is

- smooth on $[0, 1)$,
- $g|_{[0,\frac{1}{2}]} = k$,

- $g(t) = \sqrt{1-t^2}$ for t near 1, and
- $g'(t) < 0$ on $(\frac{1}{2}, 1)$.

In the standard symplectic $\mathbb{R}^4 = \mathbb{C}^2$ with polar coordinates $(r_1, \theta_1, r_2, \theta_2)$ define the domain

$$P' = \{r_1 \leq g(r_2) \ : \ r_2 \in [0,1]\} \subset P = \{r_1 \leq k \ ; \ r_2 \leq 1\}$$

contained in the polydisk P. Indeed one can think of P' as of the polydisk P with corners rounded. Denote by A' the part of the boundary of P' given by

$$A' = \left\{ r_1 = g(r_2) \ : \ r_2 \in [\frac{1}{2}, 1] \right\}.$$

Notice that A' is C^∞-tangent to ∂P near its boundary. The primitive

$$\lambda = \frac{1}{2} \left(r_1^2 d\theta_1 + r_2 d\theta_2 \right)$$

of the standard symplectic form on $\mathbb{R}^4$ restricts to A' as a contact form

$$\lambda|_{A'} = \frac{r_2^2}{2} \left(\frac{g^2(r_2)}{r_2^2} d\theta_1 + d\theta_2 \right).$$

Set $G = S^2 \times D^2$ with its standard symplectic structure $\omega_0 = \sigma_1 + \sigma_2$, where the total area of σ_1 on S^2 is 2π and the total area of σ_2 on D^2 is $k^2\pi$. Split S^2 into the northern and southern hemispheres of equal area, denoted by S^2_+ and S^2_- respectively. There exists a symplectomorphism

$$\Phi : P \to S^2_+ \times D^2 \subset G.$$

Let $H = \mathrm{cl}(G \setminus \Phi(P'))$ and $A = \Phi(A')$. So H is a 2-handle with attaching region A. Notice that $\mathrm{cl}(\partial H \setminus A)$ is fibred by symplectic disks.

Theorem 2.1.15 (Eliashberg, [6]). *Suppose* (Y, ξ) *is a contact manifold and* ω *a closed* 2*-form on* Y *with* $\omega|_\xi > 0$*. Furthermore assume we are given an open book decomposition of* Y *compatible with* ξ *with with binding* B*. Let* Y' *be obtained from* Y *via* 0*-surgery along* B *with respect to the page framing and the corresponding cobordism* X *with* $\partial X = (-Y) \sqcup Y'$*. Then* X *admits a symplectic form* Ω *such that* $\Omega|_Y = \omega$ *and* Ω *is positive on fibers of the fibration* $Y' \to S^1$*.*

Although the actual attaching of an E-handle requires one to deform the symplectic structure in a collar of the attaching region, the following corollary is implied from the construction above.

Corollary 2.1.16. *If* (H, A) *is an E-handle attached to* Y *as in the situation of Theorem 2.1.15, then the cocore of* H *has a neighborhood symplectomorphic to* $D^2 \times D^2$ *with its standard symplectic structure.*

Thus the upper boundary of the cobordism in Theorem 2.1.15 is a symplectic fibration. By this, we mean a 3-manifold Y that fibers over the circle and is equipped with a closed 2-form ω that is positive on each fiber. The line field $l = \ker(\omega)$ is transverse to the fibers. An orientation on Y and the fibers orients l, which thus can be generated by a vector field v. Picking one fiber F, the return map of this vector field is called the *holonomy* of the fibration.

In the same work, Eliashberg shows that such symplectic fibrations as obtained in Theorem 2.1.15 can be symplectically capped off. We will need the following lemma, [8].

Lemma 2.1.17. *Suppose the holonomy of a symplectic fibration* (Y, ω) *with fiber* F *is Hamiltonian. Then there is a symplectic form* Ω *on* $X = F \times D^2$ *such that* $\partial X = Y$ *and* $\Omega|_{\partial X} = \omega$.

2.2 Holomorphic curves and ruled symplectic 4-manifolds

In this section we recollect results on holomorphic curves and ruled symplectic manifolds, with a special emphasis on $S^2 \times S^2$. A self-contained reference on this material is given in [20, 25]. For a detailed account on holomorphic curves, the reader is pointed to [26].

2.2.1 Holomorphic curves in symplectic manifolds

An *almost complex structure* J on X is an automorphism $J : TX \to TX$ of the tangent bundle with $J^2 = -Id$. We say J *tamed* by a symplectic form ω on X if $\omega(v, Jv) > 0$ for all $v \in TX$, and *compatible* if furthermore $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$ for all v_1, v_2 in TX. We write $\mathcal{J}(X, \omega)$ for the set of all compatible almost complex structures. Recall that this is a nonempty and contractible set. Notice that an almost complex structure on X gives rise to a first Chern class $c_1 \in H^2(X)$ which only depends on the symplectic structure, since all compatible almost complex structures are homotopic.

Definition 2.2.1. Given a Riemann surface (Σ, j), a *holomorphic curve* in X is a map $u : (\Sigma, j) \to X$ satisfying the (non-linear, elliptic) Cauchy-Riemann equation

$$\frac{1}{2}(du \circ j - J \circ du) = 0.$$

The image of u is called an *unparametrized curve.* If Σ happens to be a sphere, a holomorphic curve is often refered to as *rational.*

We will assume that holomorphic curves are *simple*, i.e. the map u does not factor as $u' \circ p$ for some branched cover p of Riemann surfaces.

A *symplectic submanifold* $\Sigma \subset (X, \omega)$ is a surface Σ where the symplectic form ω restricts to a symplectic form on Σ. Notice that, due to the nature of the space of almost complex structures compatible with ω, given a symplectic submanifold we can always find an almost complex structure that makes it a holomorphic curve; see e.g. [27, Proposition 1.2.2].

Local properties of holomorphic curves translate from the complex point of view, i.e. the case where J is integrable. Two of them should be emphasized here. First, holomorphic curves intersect positively:

Theorem 2.2.2 (positivity of intersections)**.** *Suppose Σ and Σ' are holomorphic curves in (X, ω, J). If they do not share a component, then $[\Sigma] \cdot [\Sigma'] \geq 0$, with equality if and only if they are disjoint.*

Second the adjunction formula which, in particular, gives a homological criterion for curves to be embedded. We state it here only for rational curves.

Theorem 2.2.3 (adjunction formula)**.** *Suppose* Σ *is a rational curve in* (X, ω, J)*. Then*

$$\langle c_1(X), [\Sigma] \rangle \leq 2 + [\Sigma]^2,$$

with equality if and only if Σ *is embedded.*

Turning to a global point of view, we describe some of the properties of the space of holomorphic curves in a given homology class. This space is, as the solution space of an elliptic system, finite dimensional, at least when J is generic. If this space is nonempty then it is either compact, or can be compactified by so called cusp-curves, via Gromov compactness theorem. We will need the following 4-dimensional version, proved by Hofer, Lizan and Sikorav, [18].

Theorem 2.2.4. *Suppose* Σ_g *is a holomorphic curve in* (X, ω) *and* $\langle c_1(X), [\Sigma_g] \rangle > 0$*, then the space of unparametrized holomorphic curves near* Σ_g *is a manifold of dimension* $2(\langle c_1(X), [\Sigma_g] \rangle - 1 + g)$*, for every* $J \in \mathcal{J}$*.*

2.2.2 Ruled symplectic manifolds

A symplectic manifold (X, ω) is called *minimal* if it does not contain an embedded symplectic sphere Σ of square -1. Such a sphere is called *exceptional*. It was shown by McDuff that such a sphere has a neighborhood N_ϵ, whose boundary $(\partial N_\epsilon, \omega)$ can be identified with the boundary $(\partial D^4(\epsilon + \tilde{\epsilon}), \omega_0)$ of the ball of radius $\epsilon + \tilde{\epsilon}$ in $\mathbb{C}^2$ where $\pi \tilde{\epsilon}^2 = \omega(\Sigma)$ and $\epsilon > 0$ is sufficiently small. Hence such a curve can be *blown down* by taking out N_ϵ and replacing it with $D^4(\epsilon + \tilde{\epsilon})$. Notice that the resulting manifold is independent of ϵ (but the symplectic structure does depend on ϵ). The reverse process is called a *blow up*.

Theorem 2.2.5 (McDuff, [24]). *Every symplectic 4-manifold (X, ω) covers a minimal symplectic manifold (X', ω') which is obtained from X by blowing down a finite collection of disjoint exceptional spheres. Furthermore, given this finite collection, the symplectic form ω' on X' is unique up to isotopy.*

This also holds in a relative version, for (X, C), where C is a symplectically embedded compact 2-manifold. Such a pair is called *minimal*, if $X \setminus C$ is minimal. Then,

Theorem 2.2.6 (McDuff, [24]). *Every symplectic pair (X, C, ω) covers a minimal symplectic pair (X', C, ω') which is obtained from X by blowing down a finite collection of disjoint exceptional spheres in $X \setminus C$. Furthermore, given this finite collection, the symplectic form ω' on X' is unique up to isotopy (rel C).*

A symplectic manifold (X, ω) is called *ruled* if it is the total space of a fibration whose fibers are 2-spheres on which the symplectic form does not vanish. In case the base of this fibration is a 2-sphere, the manifold is called *rational*. Topologically there are only two such manifolds: $S^2 \times S^2$ and $S^2 \tilde{\times} S^2 \approx \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.

Theorem 2.2.7 (McDuff, [24]). *Let (X, C, ω) be a minimal symplectic pair where C is a symplectic sphere with self-intersection $[\Sigma]^2 = p \geq 0$. Then (X, ω) is symplectomorphic either to $\mathbb{CP}^2$ with its usual Kähler form or to a symplectic S^2-bundle over* *a compact surface B. Furthermore this symplectomorphism may be chosen so that it takes Σ either to a complex line or quadric in $\mathbb{CP}^2$, or to a fiber of the S^2-bundle, or (if $B = S^2$) to a section of this bundle.*

Specifically, we will need the following corollary.

Corollary 2.2.8. *Suppose (X, ω) is a symplectic manifold containing a symplectically embedded sphere Σ with self-intersection 0. Suppose that there exists an embedded sphere S with self-intersection 0, that intersects Σ geometrically once. Then (X, ω) is symplectomorphic to a blow up of $S^2 \times S^2$.*

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Chapter 3

Constructing Planar Open Books

A *plumbing tree* P is a tree where each vertex is endowed with an integer. Replacing each vertex by an unknot such that two are linked exactly once if and only if there is an edge between the corresponding vertices, gives a link $L \subset S^3$. A 3-manifold Y is obtained from a plumbing tree by doing surgery along L where the surgery coefficient on each component is the integer on the corresponding vertex.

In this chapter we construct planar open books for 3-manifolds arising from such a plumbing diagram, with the condition that the surgery coefficient is at most minus the valence of the corresponding vertex. Because these open books will have positive monodromy, they support Stein fillable contact structures. Although not used explicitly, the construction of such open books is inspired by the algorithm presented in [3] that describes how to turn rational contact surgery into a sequence of (± 1)-contact surgeries.

We will gradually build up this construction, starting with the easiest version. Plumbing on a linear tree yields a Lens space and one can obtain all contact structures in this way. Second we describe open books for small Seifert fibred spaces with integral Euler number $e_0 \leq -3$. Again, all contact structures on these spaces are constructed this way. Lastly we prove Theorem 3.3.1. Since there is no general classification of contact structures on these spaces yet, we cannot say how many of the contact structures are obtained in this way.

3.1 Lens Spaces

For coprime integers $p > q \geq 1$, consider the continued fraction expansion

$$-\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_k}}} = [a_1, \ldots, a_k], \quad a_i \leq -2,\ i = 1, \ldots, k.$$

The Lens space $L(p, q)$ is obtained from a linear plumbing tree with k vertices and labels $a_1, \ldots, a_k$. Notice that, performing a sequence of slam-dunks starting with the rightmost component, one obtains a rational surgery diagram, $\left(-\frac{p}{q}\right)$-surgery along an unknot. Instead, we will perform handle slides. Starting with the leftmost component K_1, slide the right neighbor K_2 over K_1. Thus K_2 links K_1 exactly $t_1 = (a_1 + 1)$ times and has as surgery coefficient $b_2 = a_1 + a_2 + 2$. The string $K_3, \ldots, K_k$ remains unchanged. Now we slide K_3 over K_2. Notice that afterwards, K_3 links K_1 exactly $(a_1 + 1)$ times, K_2 is linked an additional $(a_2 + 2)$ times and

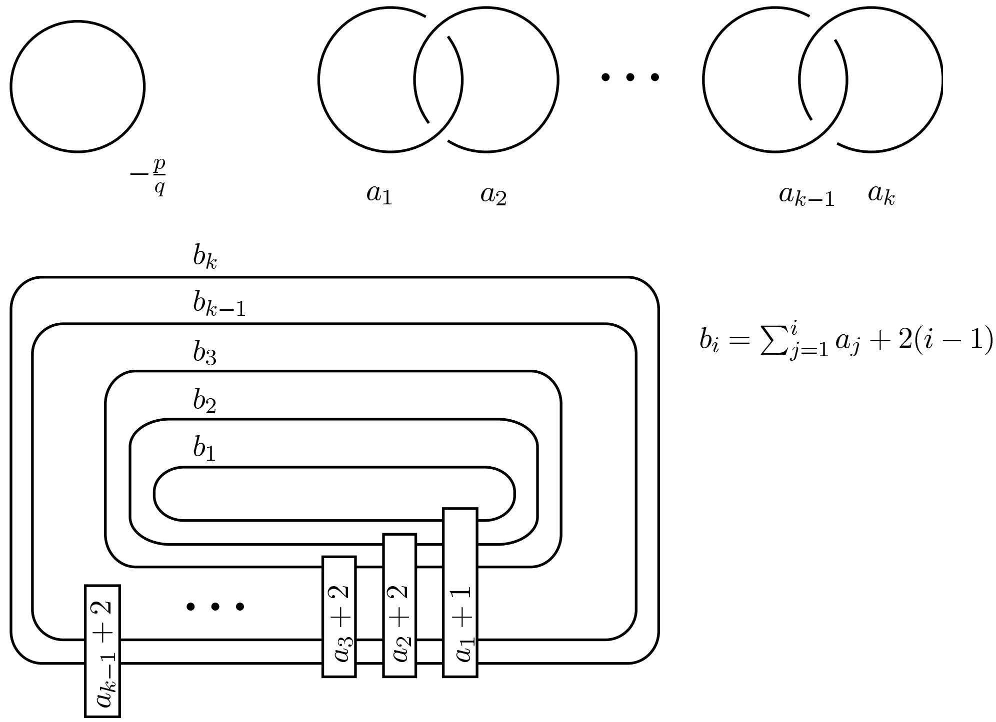


Figure 3.1: Surgery diagrams for $L(p,q)$: rational surgery, surgery on a linear tree and its rolled up version.

the surgery coefficient is $b_3 = a_1 + a_2 + a_3 + 4$. Continuing similarly, we obtain a new surgery diagram as in Figure 3.1. We call a sequence of handle slides as just performed *rolling up* a linear tree. For an explicit example, see Figure 3.2 below.

The classification of tight contact structures on $L(p,q)$ was found by Honda [19] and Giroux [12].

Theorem 3.1.1. *On the Lens space $L(p,q)$, there are exactly $\Pi_{i=1}^{n}|a_i+1|$ Stein fillable contact structures up to isotopy.*

Theorem 3.1.2. *Any tight contact structure on a Lens space $L(p,q)$ is supported by a planar open book.*

Proof. Suppose

$$-\frac{p}{q} = [a_1, \dots, a_k], \quad a_i \leq -2 \ i = 1, \dots, k$$

as above and K the link in S^3 obtained from the linear plumbing by rolling up. Recall that the surgery coefficient of a component K_i is $b_i = \sum_{j=1}^{i} a_j + 2(i-1)$, as in Figure 3.1.

We show that there exists an open book of S^3 such that each component K_i of K is contained in a page and homologically nontrivial. The aim is to employ Theorem 2.1.13 for each surgery along K_i. Start with the open book π_+ given by the positive Hopf fibration of S^3. Next, we need to arrange for each K_i to be contained in a page and that the page framing differs from the framing at hand by 1. We can realize K_1 stabilizing a parallel copy of the core of a page of π_+ $|a_1 + 2|$-times. Recall from subsection 2.1.4 that there is not a unique way to do so. We choose one. Proceeding by induction, suppose we have realized $K_1, \dots, K_{i-1}$. To get K_i, pick a parallel copy of K_{i-1} and stabilize it $|a_i + 2|$ times. Eventually we obtain an open book for a contact structure on $L(p, q)$. Since every stabilization is done on a connected component of the boundary of a page, starting with an annulus, the page of the resulting open book is planar. Moreover the number of choices during this construction is exactly the number of tight contact structures on $L(p, q)$. We are left to show that different choices yield different contact structures. Notice that Legendrian realizing K on a page gives in particular a Legendrian link in S^3. The contact structures can then be distinguished by using a result of Lisca-Matić

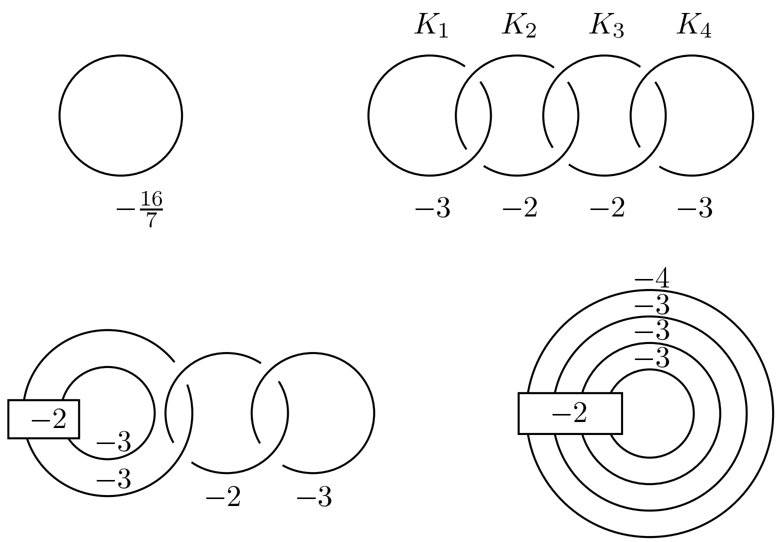


Figure 3.2: Rolling up the surgery for $L(16, 7)$.

[23]. □

Example 3.1.3. The Lens space $L(16, 7)$ is shown in Figure 3.2. Notice that $-16/7 = [-3, -2, -2, -3]$. This gives the linear plumbing tree which we roll up to obtain the last picture in Figure 3.2. We exhibit the open book for a contact structure in Figure 3.4. It is frequently useful to find a 4-manifold compatible with a given open book. Such a 4-manifold is shown in Figure 3.5. A Legendrian realization of the corresponding link K is given in Figure 3.3. Notice that this link is also obtained from contact $\left(-\frac{16}{7} + 1\right)$-surgery along a Legendrian unknot with $tb = -1$ by the algorithm described in [3].

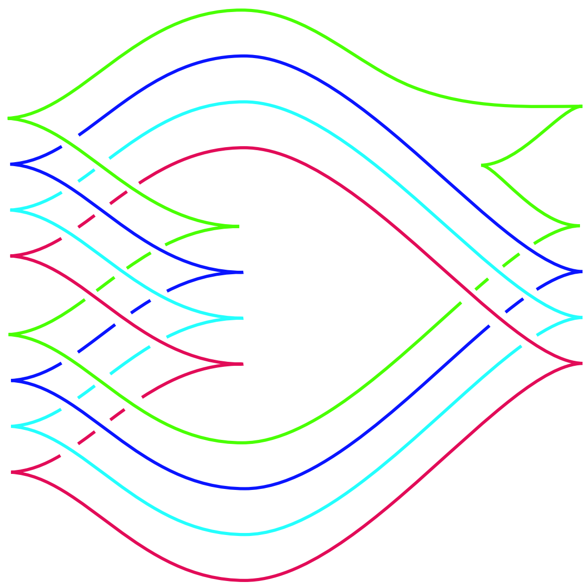


Figure 3.3: Legendrian realization of the 'rolled up' diagram in Figure 3.2.

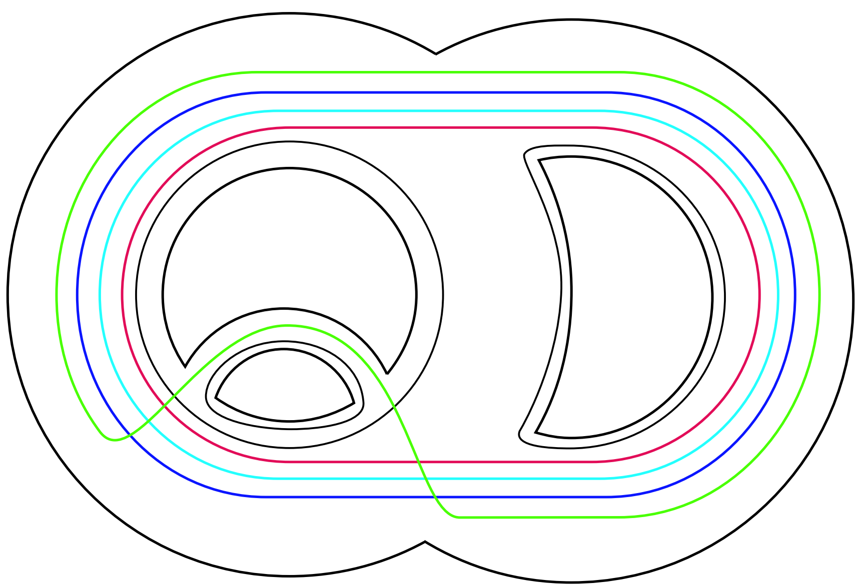


Figure 3.4: Constructing an (abstract) planar open book. The monodromy consists of one positive Dehn twists along each curve.

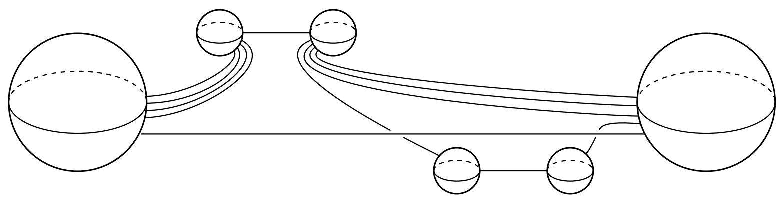


Figure 3.5: 4-dimensional picture of the planar open book. All 2-handles have framing -1.

3.2 Small Seifert fibred Spaces

A 3-manifold obtained by plumbing along a tree as in Figure 3.9 is called a *small Seifert fibred space* and denoted with $M(e_0; r_1, r_2, r_3)$, where $r_1 = [a_1, \ldots, a_{k_1}]$, $r_2 = [b_1, \ldots, b_{k_2}]$, $r_3 = [c_1, \ldots, c_{k_3}]$. The coefficient e_0 is called the *integral Euler number*. For a recollection on facts about small Seifert fibred spaces; see e.g. [14]. For small Seifert fibred spaces with $e_0 \leq -3$, the tight contact structures are classified in [34].

Theorem 3.2.1. *Suppose a Y is a small Seifert fibred space with $e_0 \leq -3$. Then any tight contact structure on Y is supported by a planar open book.*

Before giving the proof, it is illuminating to see an example. The proof will follow exactly that scheme, with more notational effort.

Example 3.2.2. Take $Y = M(-3; -3/2, -5/3, -5/3)$. This manifold has a plumbing diagram as in Figure 3.6.

First consider the surgery diagram for the Lens space given by the horizontal chain, which is $L(45, 19)$; note $-\frac{45}{19} = [-3, -2, -3, -2, -3]$. Then the third component, corresponding to the central -3, has one more stabilization than the previous

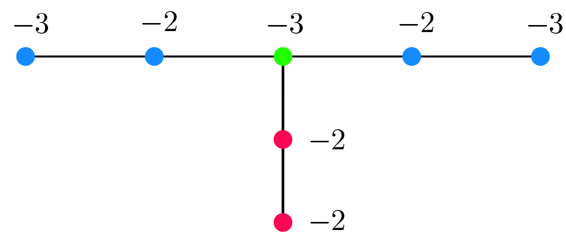


Figure 3.6: Plumbing diagram for $M(-3; -3/2, -5/3, -5/3)$.

ones. Notice that in the process of rolling up, a meridian of this component will link all its neighbors to the right exactly once. We can use such a meridian in an additional stabilization to hook in the surgery diagram for $L(3,2)$, which is the Lens space obtained from the remaining vertical chain. Figure 3.8 displays a Legendrian realization of this link.

Similar to the previous case, we can realize this link on pages of an open book S^3 with planar pages using stabilizations. Using Theorem 2.1.13 we obtain an open book for Y with planar pages, supporting a Stein fillable contact structure. Such an open book is given in Figure 3.7. Now one only needs to show that the manifold obtained in this way is indeed $M(-3; -3/2, -5/3, -5/3)$. This is done via Kirby Calculus.

proof of theorem 3.2.1. Consider a small Seifert fibred space $M(e_0; r_1, r_2, r_3)$ with $e_0 \leq -3$ and r_1, r_2, $r_3 < -1$. Denote the continued fraction expansions of r_i by $r_1 = [a_1, \ldots, a_{k_1}]$, $r_2 = [b_1, \ldots, b_{k_2}]$ and $r_3 = [c_1, \ldots, c_{k_3}]$. A plumbing diagram for M is shown in Figure 3.9.

We now produce a Legendrian surgery diagram as in the example above: Roll

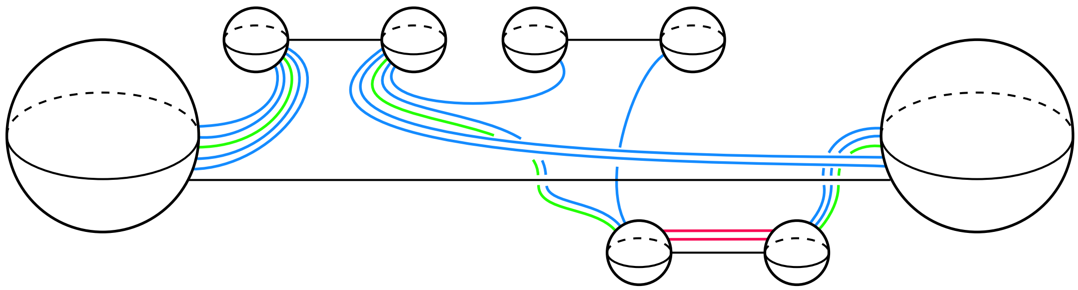


Figure 3.7: Open book for the supporting the contact structure given in Figure 3.8. All 2-handles have framing -1.

up the linear plumbing tree corresponding to the Lens space $L(p,q)$ with p, q such that

$$-\frac{p}{q} = [a_{k_1}, \ldots, a_1, e_0, c_1, \ldots, c_{k_3}]$$

to obtain a sequence knots as pushoff's of each other. Denote by A_i, B_i, C_i and E the components corresponding to a_i, b_i, c_i and e_0, respectively. Then this sequence of pushoff's starts with, say, A_{k_1}. Pick an unknot U linking the component E, and all the following ones once. We can hook in a surgery diagram of the Lensspace $L(p',q')$ with $-\frac{p'}{q'} = [b_1, \ldots, b_{k_2}]$ as follows. Roll up the linear plumbing tree for $L(p',q')$ and replace U with the link thus obtained. We can Legendrian realize this link, noticing that the component E, and all the following ones, will have one more stabilization than the previous elements.

The Legendrian surgery diagram one obtains this way yields a Stein fillable contact structure on a manifold. A planar open book supporting this contact structure is shown in Figure 3.10. Thus one only needs to see that the 3-manifold is indeed

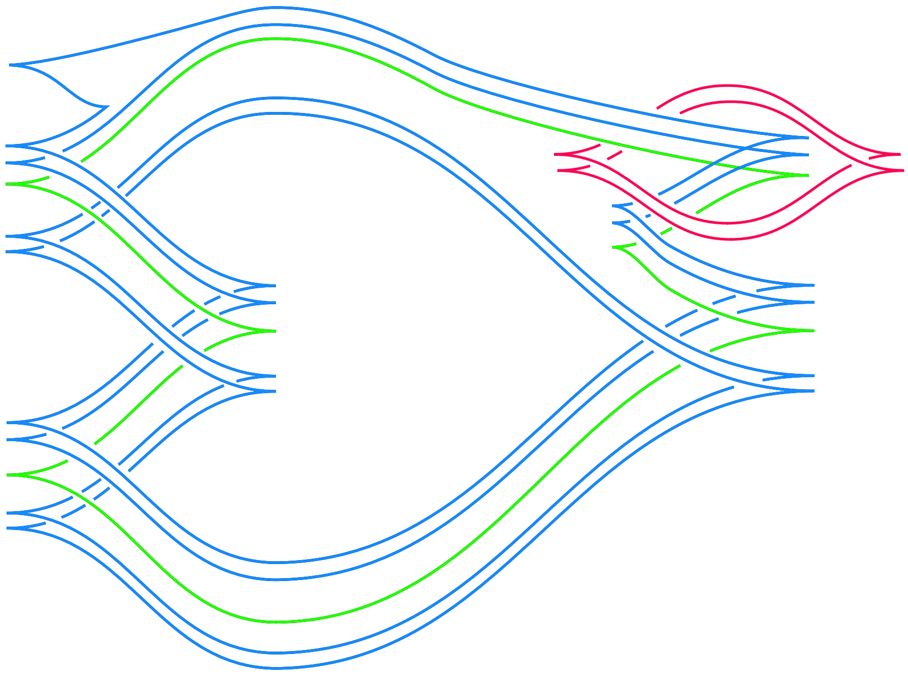


Figure 3.8: Legendrian realization of the surgery for M.

the one obtained by the plumbing graph in Figure 3.9. Set up the following notation: α_i denotes the framing of the handle A_i, similarly for β_i, γ_i and ϵ. Initially, all framings are -1. Denote by $\mathcal{A}_i$ the 1-handles used for the stabilization of A_i. Similarly $\mathcal{B}_i$ denotes the 1-handles used in the stabilization of B_i and $\mathcal{E}$ the ones used to stabilize E. First, slide B_{k_2} over the (-1)-framed 2-handles coming from the stabilization $\mathcal{B}_{k_2}$. This changes β_{k_2} to $b_{k_2}+1$ and allows to cancel the 1-handles in $\mathcal{B}_{k_2}$. Slide B_{k_2} over B_{k_2-1}. This gives $\beta_{k_2}=b_{k_2}$ and now B_{k_2} only links B_{k_2-1} once. This *untwines* B_{k_2}. One can untwine B_{k_2-1} in the same way, not affecting the properties of B_{k_2}. Inductively, one can untwine all the B_i, thereby cancelling all the 1-handles in $\mathcal{B}$. and turning the 2-handles B_i into a chain, with B_1 linking the handles in C and E once, as shown in figure 3.11.

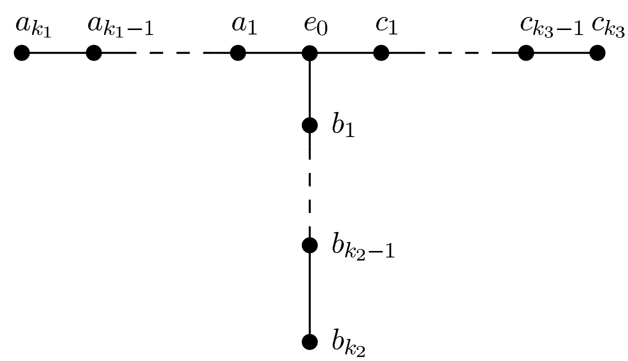


Figure 3.9: Plumbing Diagram for $M(e_0; r_1, r_2, r_3)$.

Next one can untwine successively all the handles in C, starting with C_{k_3}. This yields a chain C_1, …, C_{k_3} with framings $\gamma_i = c_i$, and C_1 links E once, as in figure 3.12.

Now untwine E. This cancels all the 1-handles in $\mathcal{E}$, changes the framing ϵ to e_0 and now E links A_1, B_1 and C_1 once.

Eventually one is able to untwine the handles in A, starting with A_1. After doing this, one has a surgery picture at hand corresponding to the plumbing diagram in Figure 3.9.

Recall that tight contact structures on these spaces were classified by Wu in [34]. Simply counting the number of possibilities for choosing stabilizations when Legendrian realizing the link obtained by rolling up and observing that different choices give rise to different contact structures (via Lisca-Matić [23]) shows that one can find such a presentation for any contact structure on $M(e_0; r_1, r_2, r_3)$. □

3.3 General plumbing trees

The proof of Theorem 3.2.1 shows that, if done in the right order, one is able to successfully untwine pushoff's of 2-handles with framing -1 to chain's of 2-handles and done in the right order already untwined chain's are not affected by further untwinings. Thus one can prove the following theorem in exactly the same way as Theorem 3.2.1.

Theorem 3.3.1. *Suppose P is a plumbing tree such that each vertex is labeled with a framing coefficient r satisfying $r \leq -d$, where d is the valence of that vertex. Then any Legendrian realization of P gives rise to a Stein fillable contact structure that is supported by a planar open book.*

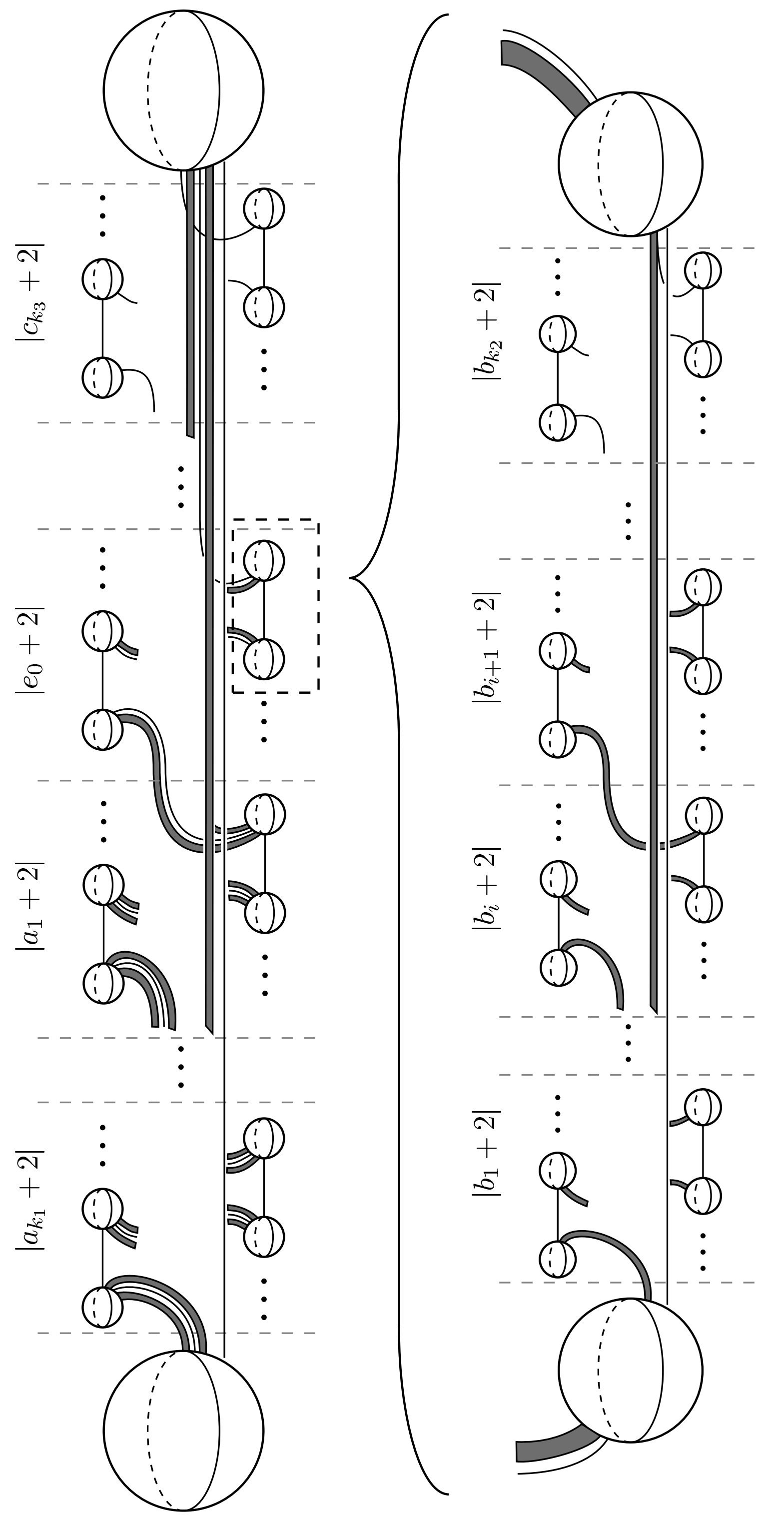


Figure 3.10: Planar open book

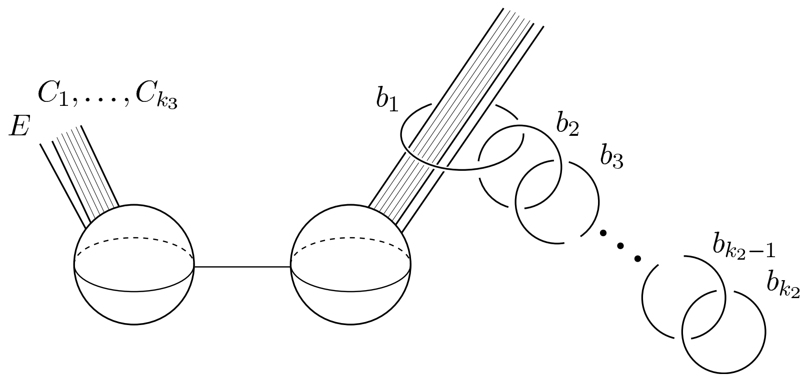


Figure 3.11: Untwining all the 2-handles B_1, …, B_{k_2}. The 1-handle belongs to $\mathcal{E}$.

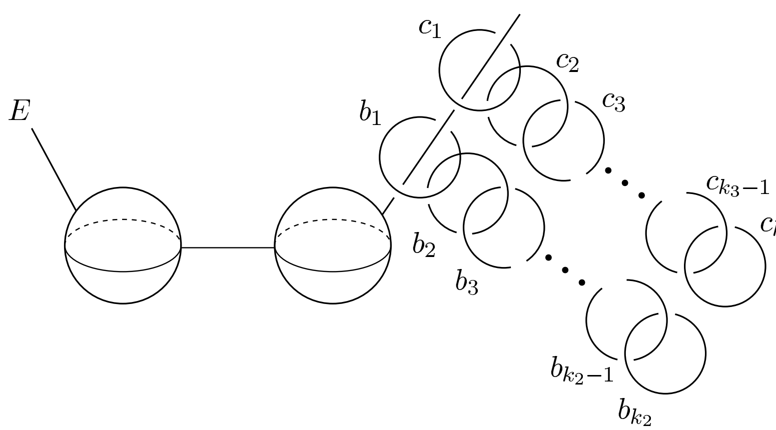


Figure 3.12: Untwining all the 2-handles C_1, …, C_{k_3}. The 1-handle belongs to $\mathcal{E}$.

Chapter 4

On the diffeomorphism types of Fillings

This chapter shows how to use planarity of open books compatible with symplectic fillable contact structures to collect information about the diffeomorphism types of their fillings.

In general, the strategy follows the line given by McDuff [24], as used by Lisca [22], Ohta-Ono [29] and others. One seeks to symplectically embed a filling into a closed symplectic manifold. Then knowing the complement of the image allows us to obtain information about the filling itself. Here, we follow Lisca [22] but use planarity of open books to produce such embeddings; see [8]. To see this, we study a concrete example.

Consider the 3-manifold Y given by the plumbing diagram as in Figure 4.1. This

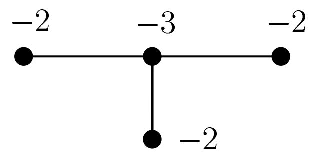


Figure 4.1: Plumbing picture for the manifold Y.

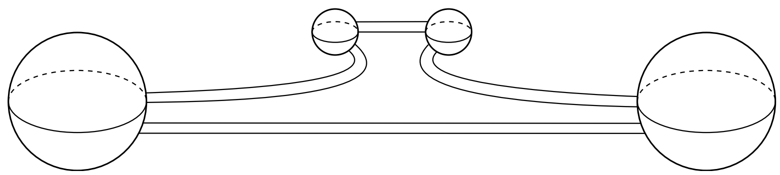


Figure 4.2: Open book compatible with (Y, ξ).

is a small Seifert fibred space with $e_0 = -3$.

From the classification of Wu [34], one concludes the following

Proposition 4.0.2. *There exist exactly two Stein fillable and hence tight contact structures on Y, up to isotopy. The two contact structures are contactomorphic.*

By Theorem 3.3.1 and its proof, one obtains immediately

Lemma 4.0.3. (Y, ξ) *is supported by the planar open book shown in Figure 4.2, where ξ is any of the two contact structures on Y.*

Figure 4.3 (a) gives a picture of the 4-manifold W as in Figure 4.2, using dotted circles for the 1-handles. Via handle slides and canceling the 1-handles one obtains (b). This verifies directly that $Y = \partial W$ is given by the plumbing in Figure 4.1. Also Figure 4.3 (c) gives a Legendrian surgery for W inducing one of the contact structures on Y. To see the other, simply rotate this picture 180°, which also proves

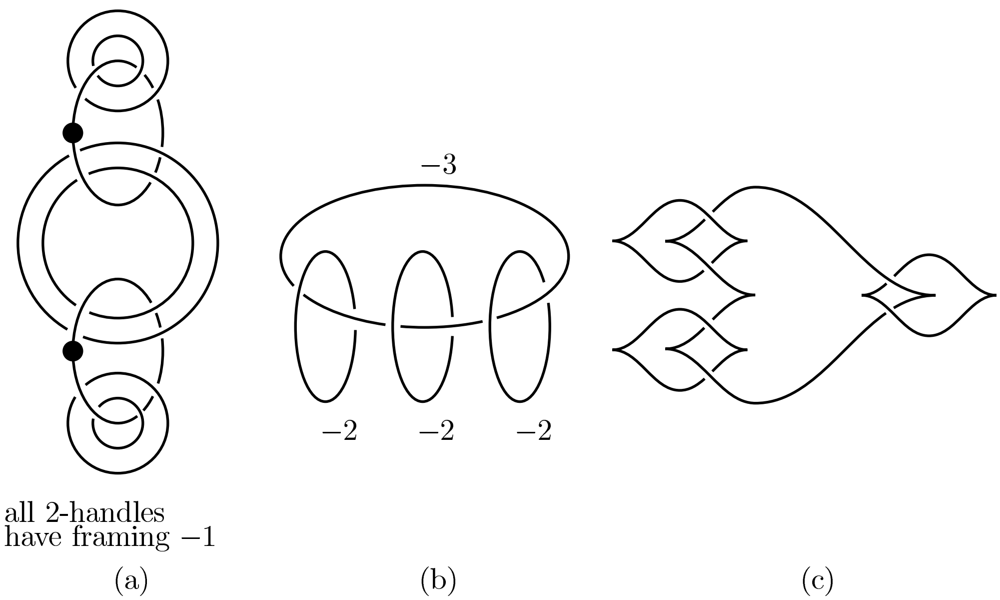


Figure4.3: Kirby pictures for W.

that these two contact structures are contactomorphic. For the remainder of this chapter, let ξ denote one of the two contact structures on Y.

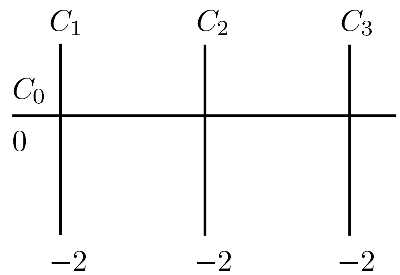
The aim of this chapter is to prove the following result:

Theorem 4.0.4. *Any symplectic filling W' for (Y, ξ) is diffeomorphic to a smooth blowup of W, obtained from the plumbing; see Figure 4.3 (c). In particular, there is a unique Stein filling of (Y, ξ) up to diffeomorphism.*

The proof takes two steps. In the first step we compactify W' to a closed symplectic manifold X and study its complement in X. In the second step we show that the diffeomorphism type of the complement of such a configuration in X is unique.

4.1 Compactification of Fillings

Theorem 4.1.1. *Suppose W' is a symplectic filling for (Y, ξ). Then, for some integer $N \geq 1$, W' is diffeomorphic to the complement of a symplectic configuration $\Gamma = C_0 \cup C_1 \cup C_2 \cup C_3$*



in $X_N = (S^2 \times S^2 \# N\overline{\mathbb{CP}}^2, \omega)$ with ω a blow up of a symplectic structure on $S^2 \times S^2$.

Proof. From Lemma 4.0.3, the contact structure ξ on Y induced by W' is supported by an open book with page F a pair of pants and monodromy ϕ consisting of two positive Dehn twists about three curves, each parallel to one of the boundary components, as in Figure 4.2. We attach Eliashberg handles H_i, $i = 1, 2, 3$, one to each of the boundary components and extend ϕ by the identity over the resulting 2-sphere, still calling it ϕ.

Further notice that ϕ is isotopic to the identity. Thus by adding these handles, we obtain $W' \subset W''$ with $\partial W'' = S^2 \times S^1$.

We can symplectically cap off W'' with a $S^2 \times D^2$. Notice that the resulting closed symplectic manifold contains an embedded symplectic sphere $S_0 = S^2 \times \{p\} \subset S^2 \times D^2$ with self-intersection 0. Thus from McDuff's theorem we conclude that (X, ω) is a blow up of a ruled surface.

Furthermore the cocore of an E-handle H_i is a symplectic disk with a neighborhood symplectomorphic to $D^2 \times D^2 \subset \mathbb{R}^4$ with its standard symplectic structure. Thus these cocores can be glued to $\{pt\} \times D^2$ in the final cap to form a symplectic sphere S_i. Each S_i is disjoint from S_j for $i \neq j$ and intersects S_0 geometrically once. Thus we obtain a symplectic configuration as in Γ and we are left to verify the self-intersection of S_i, $i = 1, 2, 3$.

To find these, we only need to find the self-intersections topologically, which can be done via Kirby calculus: start with Figure 4.2, i.e. the open book for (Y, ξ). Adding the E-handles amounts to attaching 2-handles with framing 0 with respect to a page of the open book. Then switching to dotted circle notation gives Figure 4.4 (a). Specify the boundary curves of the cocores, which are denoted by dashed circles in Figure 4.4, and endow them with labels 0. Now following through the Kirby moves to see the $S^2 \times S^1$ at hand, observe what the labels for the cocores become and these correspond to the self-intersections of the S_i. To do this, first slide each pair of (-1)-framed 2-handles over the 0-framed 2-handles to which they are parallel. Thus these only link the dashed circle specified on that 0-framed 2-handle once and blowing them down rises label of each dashed circle to 2, as in Figure 4.4 (b). Now slide each 0-framed 2-handle over its neighbors to the right, which gives Figure 4.4 (c). Now we can cancel the 1-handles, obtaining (d). From the labels of the dashed circles we read off the self-intersection number of each S_i, which is -2. Notice the change in sign that comes from the fact that we need to turn the

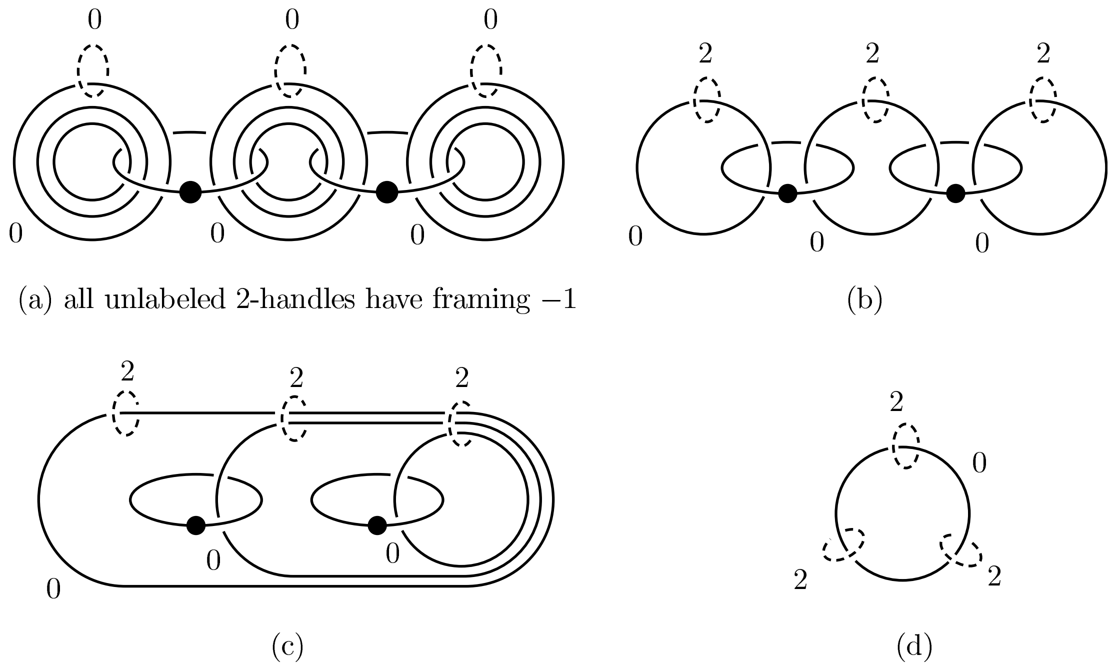


Figure4.4: Calculating the self-intersections of S_i.

handlebody upside down to see the configuration for what it is.

Because the sphere $\tilde{S}$ in the homology class $S_1 + S_0$ has self intersection 0 and intersects S_0 exactly one time, the neighborhood of $S_0 \cup \tilde{S}$ is a punctured $S^2 \times S^2$ and McDuff's result implies that (X, ω) is symplectomorphic to $(S^2 \times S^2 \# N\overline{\mathbb{CP}}^2, \omega)$ with ω a symplectic structure on $S^2 \times S^2$ blown up. Furthermore McDuff tells us that we can choose this symplectomorphism to map the sphere S_0 to $S^2 \times \{pt\}$. $\square$

4.2 Complements of the symplectic configuration

In this section we study the complement of the symplectic configuration Γ obtained in theorem 4.1.1. We do this first on the homology level.

4.2.1 Homological properties of the configuration Γ

Recall that $H_2(S^2 \times S^2 \# N\overline{\mathbb{CP}}^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \oplus_{i=1}^N \mathbb{Z}$ with intersection form

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus (-\mathbb{I}_N),$$

where $(-\mathbb{I}_N)$ denotes the negative identity $N \times N$ matrix. We fix a basis $s_1 = [S^2 \times \{pt\}]$, $s_2 = [\{pt\} \times S^2]$ and f_j, $j = 1, \dots N$, for the homology classes generated by the $\overline{\mathbb{CP}}^2$. Pick an almost complex structure J on X_N such that Γ consists of holomorphic spheres. Expressing the $[C_i]$ in terms of this basis

$$[C_0] = s_1 \tag{4.2.1}$$

$$[C_i] = \sigma_i^1 s_1 + \sigma_i^2 s_2 + \sum_{j=1}^N \phi_i^j f_j, \; i = 1, \dots, 3 \tag{4.2.2}$$

we conclude first from $[C_0] \cdot [C_i] = 1$ that $\sigma_i^2 = 1$ for all i. Thus for the following we write σ_i for σ_i^1.

We find that, since $[C_i]^2 = -2$,

$$2\sigma_i + 2 - \sum_j (\phi_i^j)^2 = 0. \tag{4.2.3}$$

From the adjunction formula $\langle c_1(X_N), [C_i] \rangle = 2 + [C_i]^2$ we learn that

$$2\sigma_i + 2 + \sum_j \phi_i^j = 0. \tag{4.2.4}$$

Subtracting equation (4.2.3) from equation (4.2.4) gives

$$\sum_j \phi_i^j + (\phi_i^j)^2 = 0. \tag{4.2.5}$$

Furthermore, since for $i \neq k$ we have $[C_i] \cdot [C_k] = 0$,

$$\sigma_i + \sigma_k - \sum_j \phi_i^j \phi_k^j = 0. \tag{4.2.6}$$

Now either Equation (4.2.3) or (4.2.4) imply

$$\sigma_i \geq -1,\ i = 1, \ldots, 3. \tag{4.2.7}$$

From (4.2.5) we conclude that

$$\phi_i^j \in \{-1, 0\},\ i = 1, \ldots, 3;\ j = 1, \ldots, N. \tag{4.2.8}$$

Then, either of the Equations (4.2.3) or (4.2.4) imply that for each i there exist

$$j_1^i, \ldots, j_{2(\sigma_i+1)}^i \in \{1, \ldots, N\}$$

such that $\phi_i^j = -1$ if and only if $j = j_l^i$ for some $l = 1, \ldots, 2(\sigma_i + 1)$.

To meet Equation (4.2.6), one first notices that $\sigma_i + \sigma_k \geq 0$. Furthermore, among the $\{j_l^i\}_{l=1}^{2(\sigma_i+1)}$ and $\{j_l^k\}_{l=1}^{2(\sigma_k+1)}$ exactly $\sigma_i + \sigma_k$ are equal. But for this to be possible, one needs that $\sigma_i + \sigma_k \leq \min\{2(\sigma_i + 1), 2(\sigma_k + 1)\}$.

Without loss of generality, we can order the $[C_i]$ such that $\sigma_1 \leq \sigma_2 \leq \sigma_3$. Then we summarize the calculations above in the following lemma.

Lemma 4.2.1. *In the situation above, there are the following possibilities for the homology classes of* $\Gamma = C_0 \cup C_1 \cup C_2 \cup C_3 \subset X_N$.

$\sigma_1 = -1$.

$$\begin{aligned}
[C_0] &= s_1 \\
[C_1] &= -s_1 + s_2 \\
[C_2] &= s_1 + s_2 - \sum_{l=1}^{4} f_{j_l^2} \\
[C_3] &= s_1 + s_2 - \sum_{l=1}^{4} f_{j_l^3}
\end{aligned}$$

without loss of generality we can assume that $f_{j_l^2} = f_{j_l^3}$ if and only if $l = 1, 2$. Notice that $N \geq 6$.

$\sigma_1 = n$ with $n \geq 0$. Then $\sigma_2 = n + s$ and $\sigma_3 = n + t$ where we can assume that $s \leq t$ with $s, t \in \{0, 1, 2\}$.

$$\begin{aligned}
[C_0] &= s_1 \\
[C_1] &= ns_1 + s_2 - \sum_{l=1}^{2(n+1)} f_{j_l^1} \\
[C_2] &= (n+s)s_1 + s_2 - \sum_{l=1}^{2(n+s+1)} f_{j_l^2} \\
[C_3] &= (n+t)s_1 + s_2 - \sum_{l=1}^{2(n+t+1)} f_{j_l^3}
\end{aligned}$$

Furthermore notice that

$$\left| \{j_l^i\}_{l=1}^{2(\sigma_i+1)} \cap \{j_l^k\}_{l=1}^{2(\sigma_k+1)} \right| = \sigma_i + \sigma_k, \; i \neq k$$

and

$$\max\{0, 2n + s + t - 2\} \leq \left| \bigcap_{i=1}^{3} \{j_l^i\}_{l=1}^{2(\sigma_i+1)} \right| \leq 2n + s.$$

Thus there are 10 subcases for $\sigma_1 = n \geq 1$ and 7 in case $\sigma_1 = 0$.

4.2.2 Blowing down to a minimal model

We know from McDuff [24] that one can always blow down symplectic spheres with square -1 and hence obtain a minimal symplectic manifold. This is also possible relative to a symplectic configuration. The following lemma generalizes verbatim from [22, Lemma 4.5]. We provide the argument here for completeness.

Lemma 4.2.2. *In the situation of theorem 4.1.1, let J be an almost complex structure tamed by ω such that Γ is holomorphic. Then, there exists a holomorphic sphere $S \subset X_N$ with square -1 and $[S] \cdot [C_0] = 0$. Furthermore, there exists such a S disjoint from Γ if and only if there exists a symplectic sphere S of square -1 such that $[S] \cdot [C_i] = 0$ for $i = 1, 2, 3$.*

Proof. Because X_N is obtained from $S^2 \times S^2$ by blowing up, there exists a symplectic sphere $S \subset X_N$ of square -1 such that $[S] \cdot [C_0] = 0$. By [24, Lemma 2.1] the homology class $[S]$ is either represented by an embedded sphere or a cusp-curve $S_1 \cup \ldots \cup S_l$, i.e. a union of (not necessarily embedded) holomorphic spheres. In the first case, the first part of the lemma is proved. In the second case, notice that

$$[S] \cdot [C_0] = ([S_1] + \ldots + [S_l]) \cdot [C_0] = 0$$

which, by positivity of intersection, implies that

$$[S_i] \cdot [C_0] = 0 \text{ for } i = 1, \ldots, l.$$

Therefore

$$[S_i]^2 \leq -1 \text{ for } i = 1, \ldots, l.$$

But

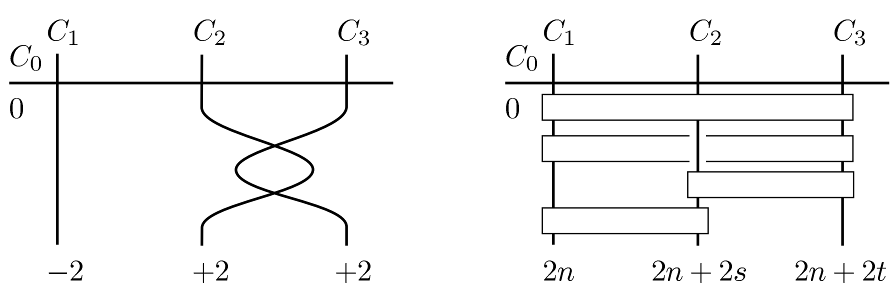
$$
\begin{aligned}
1 &= \chi(S) + [S] \cdot [S] \\
&= \langle c_1(X_N), [S] \rangle \\
&= \sum_{i=1}^{l} \langle c_1(X_N), [S_i] \rangle \\
&= \sum_{i=1}^{l} \chi(S_i) + [S_i] \cdot [S_i]
\end{aligned}
$$

implies that $[S_i] \cdot [S_i] = -1$ for at least one index $i \in \{1, \ldots, l\}$. By the adjunction formula [24] chosing $S = S_i$ is an embedded sphere.

If $[S]$ is orthogonal to all the classes $[C_i]$ then, by positivity of intersections, S must be disjoint from Γ, which proves the second part of the lemma. □

Thus, blowing down all the existing -1 spheres yields the following

Proposition 4.2.3. *There exists a sequence of blowdowns (X_N, Γ) to $(S^2 \times S^2, \Gamma')$ where Γ' is given as follows. (The boxes in the right hand figure denote the appropriate number of intersections, as in Lemma 4.2.1.)*



case $\sigma_1 = -1$ *case* $\sigma_1 = n \geq 0$

We like to argue that the symplectic configuration thus obtained is unique. For the case where $\sigma_1 \geq 0$ this follows immediately from the following

Proposition 4.2.4. *Suppose S_i, $i = 1, 2$ are embedded symplectic 2-spheres in $(S^2 \times S^2, \omega)$ for some symplectic structure ω, representing the same homology class $[S_i] = ns_1 + s_2$, such that $[S_1] \cdot [S_2] = 2n$. Here denote by s_1 and s_2 the homology classes $[S^2 \times \{pt\}]$ and $[\{pt\} \times S^2]$ in $H_2(S^2 \times S^2; \mathbb{Z})$. Then the two spheres are isotopic.*

Proof. Choose an almost complex structure that makes the spheres holomorphic. If $n < 0$ the two spheres coincide by positivity of intersections. If $n = 0$, again by positivity of intersections, the two spheres either coincide or are disjoint. In the latter case, $S_i = \{p_i\} \times S^2$ for two distinct points in the first factor. Any path on that sphere joining p_1 and p_2 provides an isotopy. In the case $n > 0$ notice that S_1 and S_2 intersect in $2n$ points (counting multiplicity). The moduli space of spheres in this class is a manifold of real dimension $2(c_1([ns_1+s_2])-1) = 2(2n+1)$. Keeping $S_1 \cap S_2$ fixed, there exists a path $\gamma : [0,1] \to S^2 \times S^2$ with $\gamma(t) \cap S_1 = \gamma(0)$, $\gamma(t) \cap S_2 = \gamma(1)$ and such that for each t there is a holomorphic sphere S_t through the $2n+1$ points $S_1 \cap S_2 \cup \gamma(t)$. This provides an isotopy from S_1 to S_2. ☐

The case where $\sigma_1 = -1$ is more difficult due to the presence of a symplectic sphere of square -2. In this case we can use a construction by Abreu [28] who shows that this case is symplectomorphic to a standard Hirzebruch surface.

4.2.3 Proof of Theorem 4.0.4

Starting with one of these unique configurations in $S^2 \times S^2$, we can blow up back to the situation in Theorem 4.1.1. Doing this in all possible ways and proving the complements of the configuration Γ thus obtained are diffeomorphic is now possible by using Kirby Calculus.

We show this process of blowing up in one situation, c.f. Figure 4.5. All other cases are obtained similarly; see Figures 4.6 and 4.7. Start with the usual handle decompostion for $S^2 \times S^2$, shown in the leftmost part of Figure 4.5. Add two canceling 2/3-handlepairs and slide the 2-handles over the 0-framed 2-handle. Thus we find three unlinked 2-handles linking once an unknot and all components have framing 0. Now pick one of these three handles and subtract it from the unknot. When adding the other two 2-handles to the unknot, we find the middle of Figure 4.5. We can blow up the crossings and then each component individually until each of the three components have framing -2. This is shown in the rightmost part of Figure 4.5. Theorem 4.0.4 follows from the following theorem.

Theorem 4.2.5. *Suppose W' is the complement of $\Gamma \subset X_N$ obtained by blowing up a minimal model from Proposition 4.2.3. Then W' is diffeomorphic to a smooth blow up of W as in Figure 4.3 (a).*

Proof. We begin by examining all possible ways to blow up a minimal model as described by Proposition 4.2.3 to get back to the original configuration $\Gamma \subset X_N$. For the case $\sigma_1 = -1$ there is only one way to do this, shown in the rightmost of

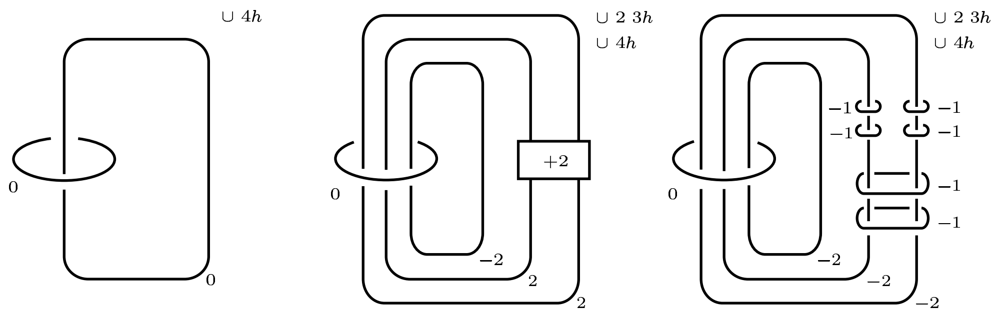


Figure 4.5: Blowing back up to Γ from the minimal model in case $\sigma_1 = -1$.

Figure 4.5. The cases where $\sigma_1 = n \geq 0$ are shown in Figure 4.6 and 4.7.

Immediately from Figures 4.5, 4.6 and 4.7 one realizes that all the complements are diffeomorphic up to blowup. Suppose there is a component, coming from the blow up procedure, that links all three (-2)-framed 2-handles once. We can slide such a component over the 0-framed 2-handle and free it from the picture, without changing its framing from -1. Thus such components can be blown down. Then, by again sliding the components coming from the blow up procedure about the 0-framed 2-handle, one can get from one picture to the other. Such handle slides do not change the diffeomorphism type of the complement.

This shows that there is at most one filling up to diffeomorphism and blowup. Since we already provided one, the theorem is proved.

We finish by explicitly showing, for one case, how to find the filling that was described in 4.3. To get a handle on $W' = X_N \setminus \text{nbhd}(\Gamma)$ we do the following. Put all the framings of 2-handles coming from Γ and the blow up circles in brackets $\langle\cdot\rangle$. Then specify cocores of the blowup circles and label them 0. Now we can use

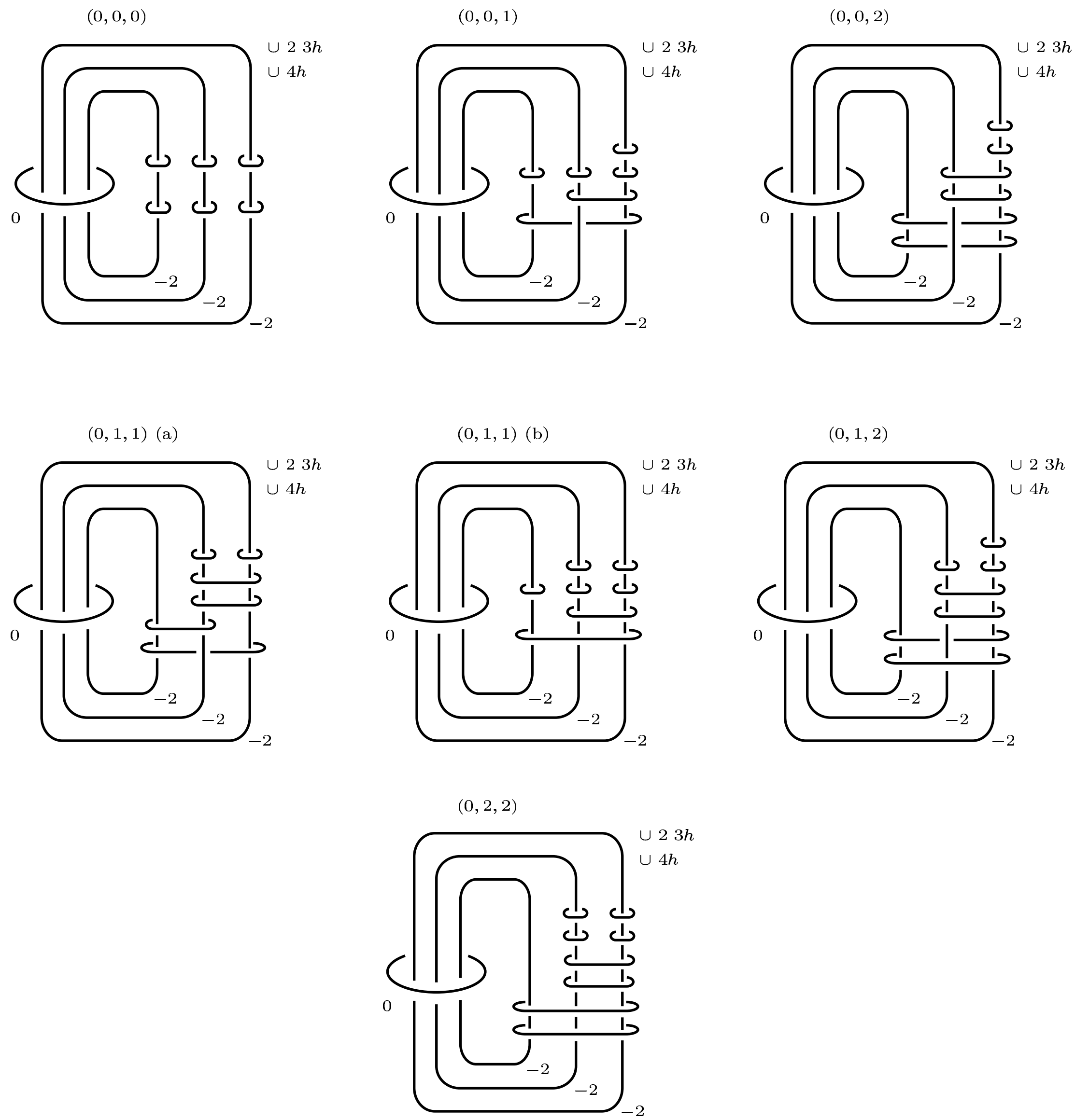


Figure 4.6: Blowing back up to Γ from the minimal model in case $\sigma_1 = 0$. (all unlabeled 2-handles have framing -1. The triple of numbers on top denotes $(\sigma_1, \sigma_2, \sigma_3)$.)

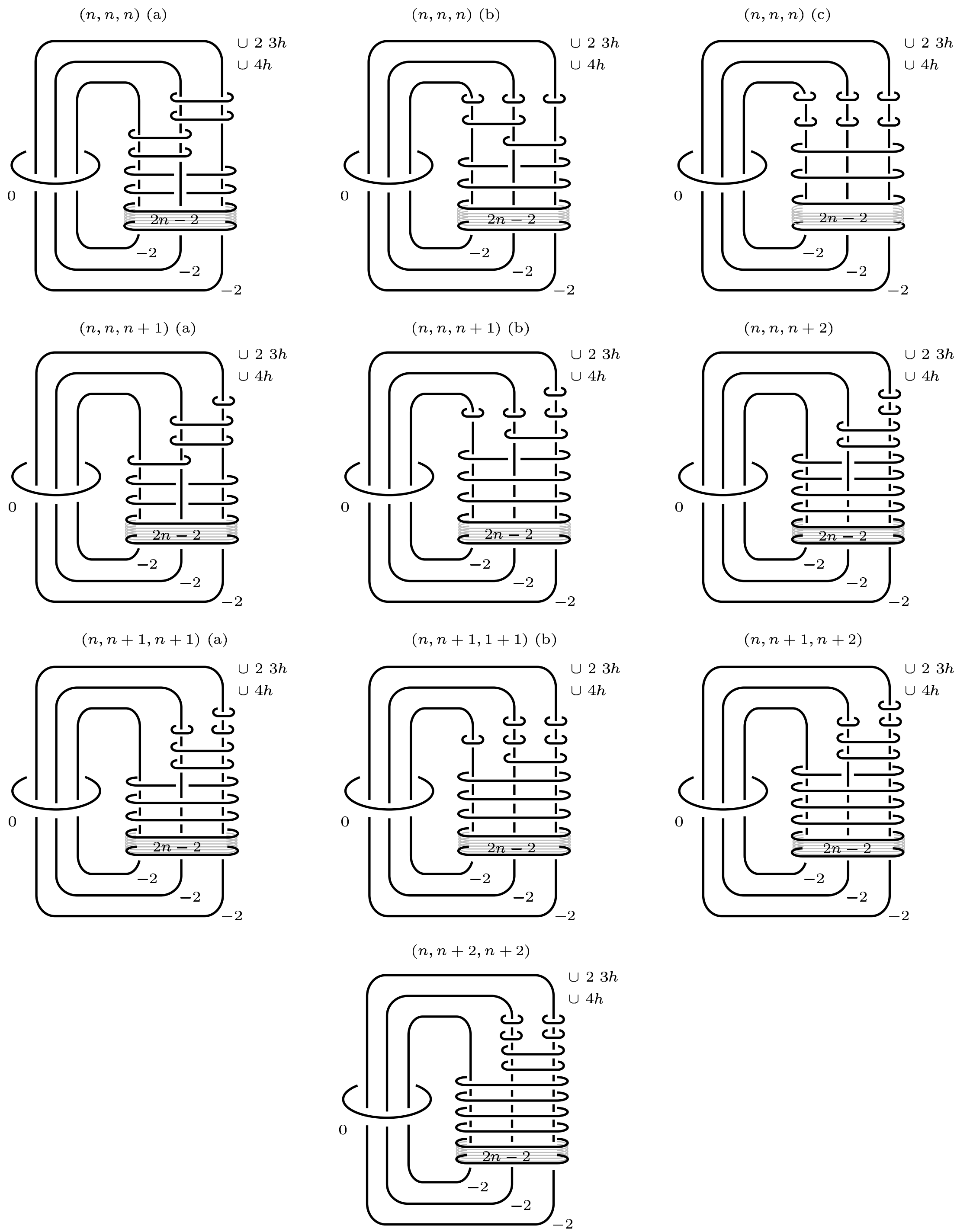


Figure 4.7: Blowing back up to Γ from the minimal model in case $\sigma_1 \geq 1$. (all unlabeled 2-handles have framing -1.)

Kirby calculus on the $\langle\cdot\rangle$-framed handles to simplify the picture. Eventually turning the handlebody upside down gives a picture of W'; see [15]. This is explained in Figure 4.8. Starting with the diagram on top, first blow down all the $\langle -1\rangle$-framed 2-handles. This gives the second diagram. Then, sliding the $\langle 0\rangle$-framed handles over its neighbors to the right yields the third diagram. In there, the two rightmost $\langle 0\rangle$-framed handles bound canceling 3-handles. When turning this handlebody upside-down those 3-handles become 1-handles and then erasing all the $\langle\cdot\rangle$-framed 2-handles gives a diagram for W, see the last picture. This is exactly what is shown already in Figure 4.3.

$\square$

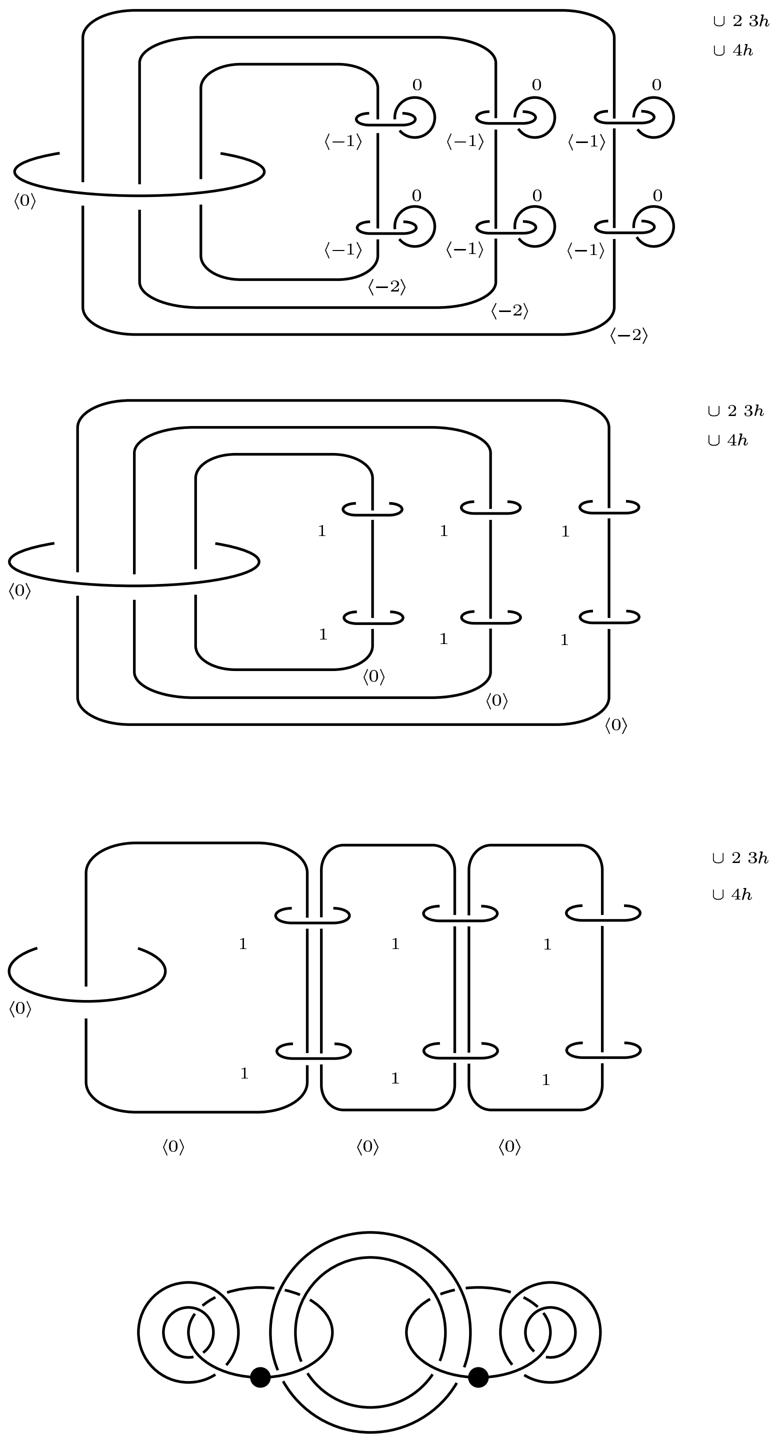


Figure 4.8: Recreating W.

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