Open ball of radius r centered at x_0

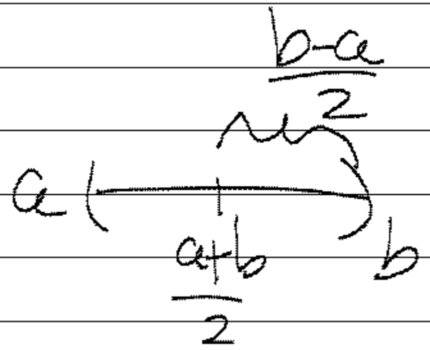
$$B_r(x_0) = \{ x \in E : d(x, x_0) < r \}$$

Closed ball of radius r centered at x_0

$$C_r(x_0) = \{ x \in E : d(x, x_0) \leq r \}$$

Example In $\mathbb{R}$

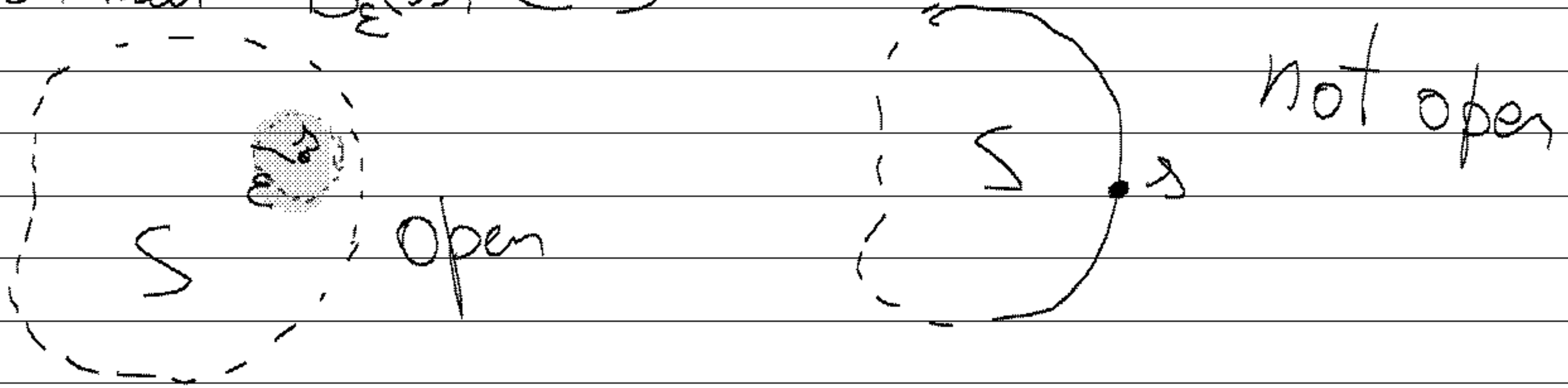
$$(a, b) = \{ x \in \mathbb{R} : a < x < b \} = B_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$$



$x_0, \varepsilon \in \mathbb{R}$ $\quad \varepsilon > 0 \quad$ $B_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon)$

Open sets

<u>Def</u>: $S \subset E$. We say that S is open if $\forall s \in S$ $\exists\, \varepsilon > 0$ such that $B_\varepsilon(s) \subset S$



<u>Prop</u>: 1) ϕ is open

2) E is open

3) Union of open sets is open

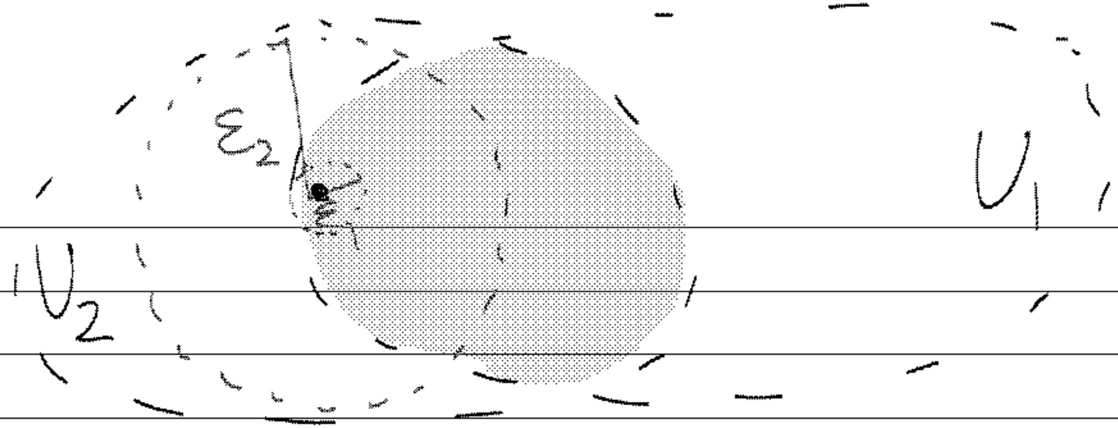
4) The intersection of any number of open sets is open

Proof: 3) Let U_α be open $\forall\, \alpha \in I$. Let $A = \bigcup_{\alpha \in I} U_\alpha$

Let $x \in A \Rightarrow \exists\, \alpha_0 \in I$ such that $x \in U_{\alpha_0}$, since U_{α_0} is open $\exists\, \varepsilon > 0$ such that $B_\varepsilon(x) \subset U_{\alpha_0} \subset A$ ✓

4) Let V_i be open for $1 \le i \le n$. Let $B = \bigcap_{i=1}^{n} V_i$. Let $x \in B$ $\Rightarrow x \in V_i$ for all $1 \le i \le n$. Since each V_i is open $\exists\, \varepsilon_i > 0$ such that $B_{\varepsilon_i}(x) \subset V_i$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\} > 0$

$B_\varepsilon(x) \subset B_{\varepsilon_i}(x) \subset V_i$ for all $1 \le i \le n \Rightarrow B_\varepsilon(x) \subset B$ ✓

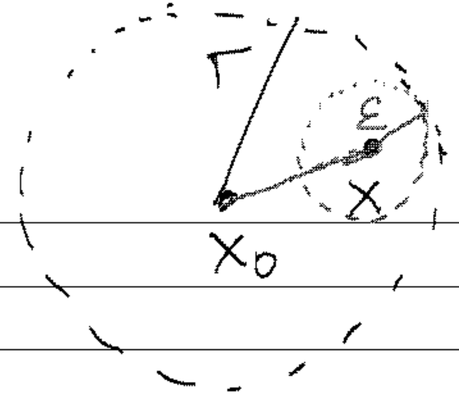


<u>Example</u>: $E=\mathbb{R}$ $\quad U_i = B_{1/i}(0) = \left(-\frac{1}{i}, \frac{1}{i}\right)$

$\bigcap_{i=1}^{\infty} U_i = \{0\}$ is not open.

<u>Prop</u>: Open balls are open

<u>proof</u>: Let $U = B_r(x_0)$. Let $x \in U$



—— $d(x_0, x)$

— ε

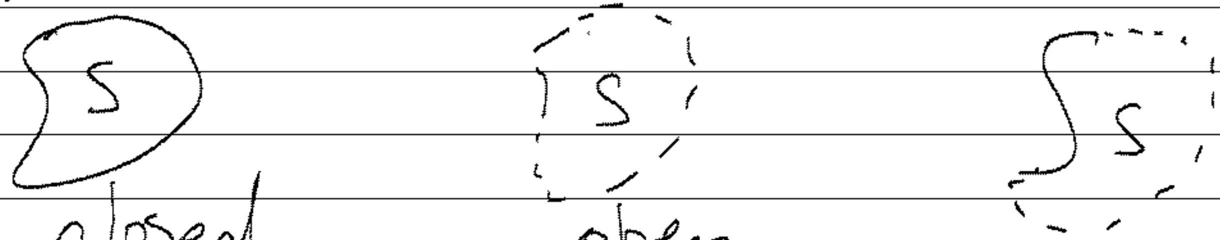
$r = d(x_0, x) + \varepsilon \qquad \varepsilon = r - d(x_0, x) > 0$

IWS that $B_\varepsilon(x) \subset B_r(x_0)$. Let $y \in B_\varepsilon(x) \Rightarrow$
I want to show

$d(x,y) < \varepsilon \Rightarrow \quad d(x_0,y) \leq d(x_0,x) + d(x,y) < d(x_0,x) + \varepsilon =$

$= d(x_0,x) + r - d(x_0,x) = r \Rightarrow y \in B_r(x_0)$ ✓

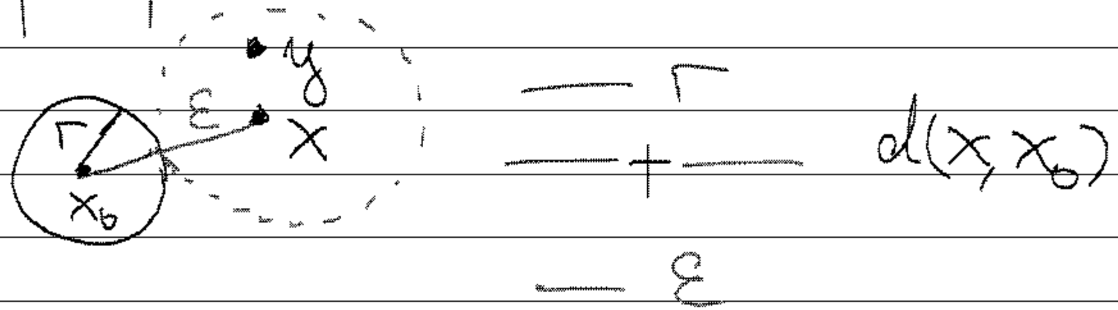
<u>Def</u>: S is closed if S^c is open



closed   open   neither

Prop: Closed balls are closed

proof Let $r>0$ $x_0 \in E$. Let $x \in (C_r(x_0))^c$



$$\varepsilon = d(x,x_0) - r$$

Let $y \in B_\varepsilon(x) \Rightarrow d(x,y) < \varepsilon$

$$d(x_0,y) + d(y,x) \geq d(x_0,x)$$

$$d(x_0,y) \geq d(x_0,x) - d(x,y) > d(x_0,x) - \varepsilon = r$$

Thus $B_\varepsilon(y) \subset \left(C_r(x_0)\right)^c$ ✓