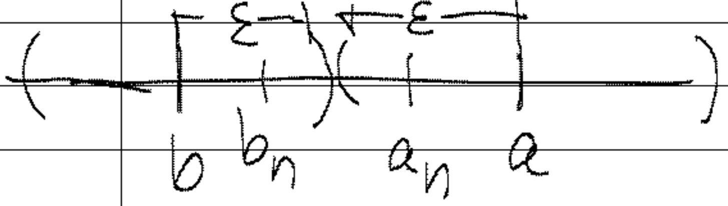
Prop: $a_n, b_n \in \mathbb{R}$ $a_n \le b_n \ \forall n$. $a_n \to a$ $b_n \to b$.

Then $a \le b$

Proof: If $b < a$. Let $\varepsilon = \frac{a-b}{2}$. Since $a_n \to a$ $\exists N_1$:



$|a_n - a| < \varepsilon$ if $n \ge N_1$. Since $b_n \to b$

$\exists N_2 : |b_n - b| < \varepsilon$ if $n \ge N_2$.

Let $n \ge \max\{N_1, N_2\} \Rightarrow$ $a_n > a - \varepsilon$ and $b_n < b + \varepsilon$

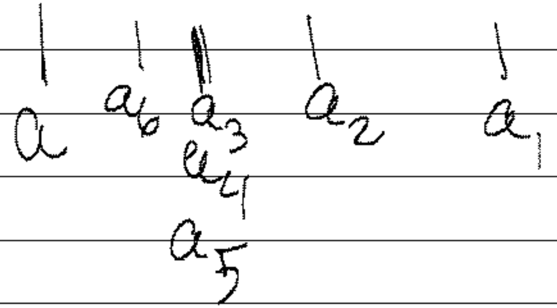
$\underbrace{a_n > \frac{a+b}{2} \qquad b_n < \frac{a+b}{2}}$

contradiction $\Leftarrow$ $a_n > b_n$ $\Longleftarrow$

Def: a_n is decreasing if $a_{n+1} \le a_n \quad \forall n$

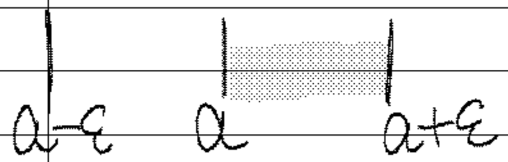
" increasing if $a_{n+1} \ge a_n \quad \forall n$

Prop: a_n is decreasing and bounded $\Longrightarrow$ it converges



proof: Let $a = glb\{a_1, a_2, \dots, a_n, \dots\}$

Let $\varepsilon > 0$. Since $a = glb\{a_n\} \Longrightarrow$ $a+\varepsilon$ is not a lb of $\{a_n\}$. $\Longrightarrow \exists N$ such that $a_N < a+\varepsilon \Longrightarrow \forall n \ge N \Longrightarrow$



$$a \le a_n \le a_N < a+\varepsilon$$

bec a is lb   because it is decreasing

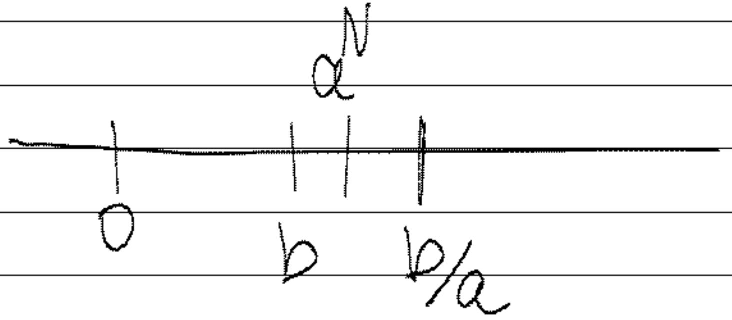
$\Rightarrow |a_n - a| < \varepsilon \quad \forall\, n \geq N.$

<u>Example</u>: $a \in \mathbb{R}$ $\quad a > 0$ & $a < 1$ $\Rightarrow \lim\limits_{n \to \infty} a^n = 0$

proof: $0 < a^{n+1} = a\, a^n < a^n$ because $a < 1$. Let $b = \lim\limits_{n \to \infty} a^n =$

$= \text{glb}\, \{a_1, \dots, a_n, \dots\}$

If $b > 0$



$b < \frac{b}{a} \Rightarrow \frac{b}{a}$ is not a lb of $\{a_1, \dots, a_n, \dots\}$

$\Rightarrow \exists N$ such that $\quad a^N < \frac{b}{a}$

$a^{N+1} < b \Rightarrow b$ is not a lb. contradiction

Completness

<u>Def</u>: $x_n \in E$. x_n is a Cauchy sequence if $\forall \varepsilon > 0$ $\exists N$ such that if $n, k \geq N$ then $d(x_k, x_n) < \varepsilon$

<u>Prop</u>: $x_n \longrightarrow x \implies x_n$ is Cauchy

<u>proof</u>: Let $\delta > 0$. $\exists N$: $d(x_n, x) < \delta$ if $n \geq N$.

Let $k, n \geq N$ $\quad d(x_k, x_n) \leq d(x_k, x) + d(x, x_n) < 2\delta$

Given $\varepsilon > 0$, select δ above as $\delta = \frac{\varepsilon}{2}$ to get the N: $k, n \geq N$

$d(x_k, x_n) < \varepsilon$ ✓