Note Title 2/17/2016

Prop: x_n Cauchy $\Rightarrow$ x_{n_k} is also Cauchy

proof: Let $\varepsilon > 0$. $\exists N$ such that $n, m \geq N$ $d(x_n, x_m) < \varepsilon$

Let K such that $n_K \geq N \Rightarrow$ if $k, l \geq K \Rightarrow n_k \geq n_K \geq N$

$n_l \geq n_K \geq N$

$\Rightarrow d(x_{n_l}, x_{n_k}) < \varepsilon$

Prop: x_n Cauchy $\Rightarrow$ x_n bounded

proof: $\exists N$ such $n, m \geq N$ $d(x_n, x_m) < 1$

$\{x_n\}_{n \in \mathbb{N}} = \{x_1, x_2, \dots, x_n, \dots\} \subset \{x_1, x_2, \dots, x_{N-1}\} \cup B_1(x_N) \subset$

$\subset B_{R+1}(x_N)$

Let $R = \max\{d(x_1, x_N), d(x_2, x_N), \ldots, d(x_{N-1}, x_N), 1\}$

<u>Prop</u>: x_n Cauchy. x_{n_k} converges $\Longrightarrow$ x_n converges

<u>proof</u>: Let $\varepsilon > 0$ $\quad x_{n_k} \longrightarrow x \Rightarrow \exists K: k \geq K \Rightarrow d(x_{n_k}, x) < \frac{\varepsilon}{2}$ $\quad$ **

$\exists N: n, m \geq N \Rightarrow d(x_n, x_m) < \frac{\varepsilon}{2}$ $\quad$ *

Let k_0 such that $k_0 \geq K$ and $n_{k_0} \geq N$

If $n \geq N$ then

$$d(x_n, x) \leq d(x_n, x_{n_{k_0}}) + d(x_{n_{k_0}}, x) < \varepsilon \qquad \checkmark$$

$< \varepsilon/2$ because of (*) $\qquad < \varepsilon/2$ because of (**)

<u>Def</u>: A metric space E is said to be complete if every Cauchy sequence converges

<u>Ex</u>: $E = (0,1) \qquad d(x,y) = |x-y|$

$x_n = \frac{1}{n}$ is Cauchy but does not converge in E

<u>Prop</u>: E complete. $S \subset E$. S closed. Then, S is complete.

<u>proof</u>: x_n Cauchy in $S \Rightarrow x_n$ is Cauchy in $E \Rightarrow x_n \to x$ for some $x \in E \Rightarrow$ since S is closed, $x \in S$.

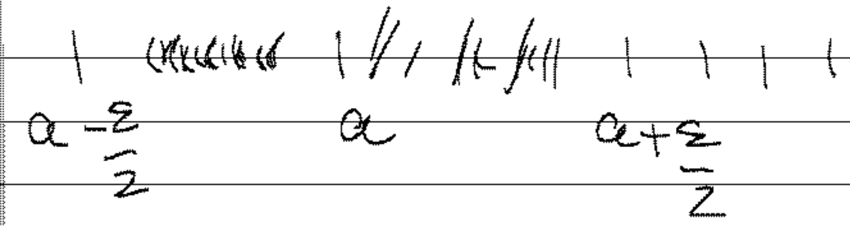
Th: $\mathbb{R}$ is complete

proof: x_n Cauchy.

$S = \{ x \in \mathbb{R} : \ x \le x_n \text{ for an infinite number of } n \}$

$\{x_n\}_{n \in \mathbb{N}}$ is bounded $\Rightarrow$ S is also bounded. Let $a = \text{lub } S$

Let $\varepsilon > 0$. $\exists N : n, k \geq N \Rightarrow |x_n - x_k| < \frac{\varepsilon}{2}$ (because x_n is Cauchy)



$a - \frac{\varepsilon}{2}$ is not an ub of S $\Rightarrow \exists$ $a - \frac{\varepsilon}{2} < x \le a$ such that $x \in S$.

$\Rightarrow$ $x \le x_n$ for an ∞ number of n.

$a+\frac{\varepsilon}{2}$ is not in $S \Rightarrow \exists$ M such that $n \geq M \Rightarrow$ $x_n \leq a+\frac{\varepsilon}{2}$

$\exists$ an ∞ # of n such that $|x_n - a| \leq \frac{\varepsilon}{2}$

Let $n \geq N$. Let $k \geq N$ and $|x_k - a| \leq \frac{\varepsilon}{2}$

$$|x_n - a| \leq \underbrace{|x_n - x_k|}_{<\frac{\varepsilon}{2}} + \underbrace{|x_k - a|}_{\leq \frac{\varepsilon}{2}} < \varepsilon \quad \checkmark$$

<u>Obs</u>: $x_k \in \mathbb{R}^n$ $\quad x_k = (x_{k1}, x_{k2}, \dots, x_{kn})$

1) x_k is Cauchy $\Longleftrightarrow$ x_{ki} is Cauchy for all $1 \leq i \leq n$

prove it

2) $x_k \longrightarrow x \iff x_{ki} \longrightarrow x_i$ for all $1 \leq i \leq n$

prove it

<u>Corollary</u>: $\mathbb{R}^n$ is complete.

<u>Compactness</u>

<u>Def</u>: $S \subset E$. E metric space. S is compact if:

$S \subset \bigcup_{i \in I} U_i$ with each U_i open, then $\exists\ i_1, \ldots, i_r \in I$

such that $S \subset \bigcup_{j=1}^{r} U_{i_j}$

Example: 1) Any finite set is compact

$$\{x_1, \dots, x_r\} \subset \bigcup_{i \in I} U_i$$

For each x_ℓ $\exists\, i_\ell$ such that $x_\ell \in U_{i_\ell}$

then $\{x_1, \dots, x_r\} \subset \bigcup_{\ell=1}^{r} U_{i_\ell}$