<u>Obs</u>: X is infinite $\Leftrightarrow \exists\, X' \subsetneq X$ and

$f: X \to X'$ ~~onto~~ one onto.

<u>proof</u>: ~~f~~ X finite $\Rightarrow X' = X$ if $X' \subset X$

& $f: X \to X'$ is one to one onto

We proved $\Leftarrow$)

$\Rightarrow$) Assume X is infinite. I want to prove

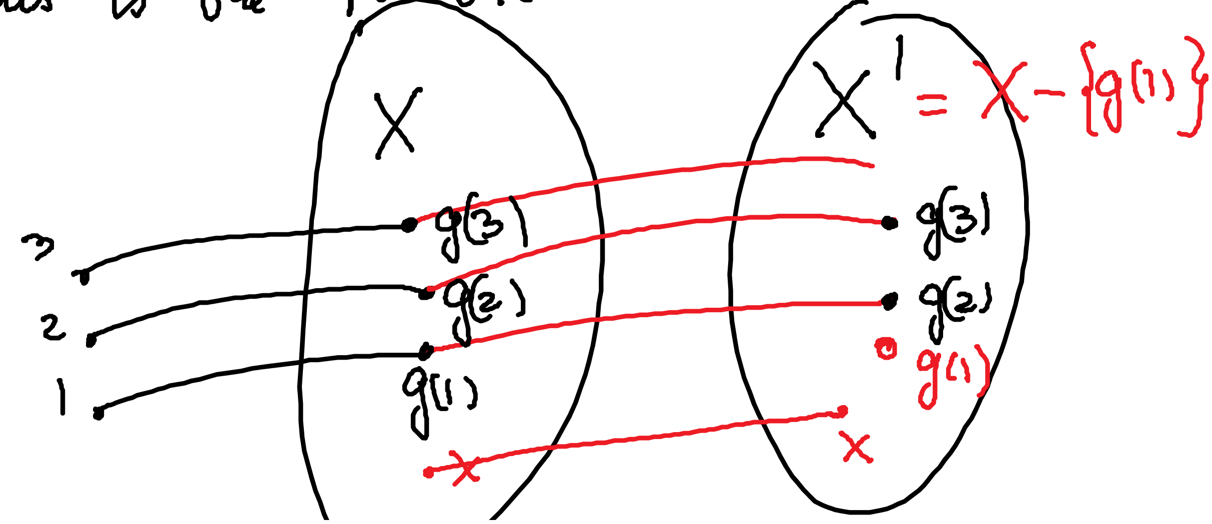
that such X' exists

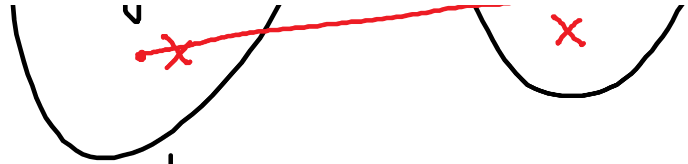
X is infinite. $\quad \mathbb{N} \xrightarrow[\text{one to one}]{g} X \xrightarrow[f]{} X'$

$$X' = X - g(1)$$

$$f(x) = \begin{cases} g(n+1) & \text{if } x = g(n) \\ x & \text{if } x \neq g(n) \text{ for all } n \end{cases}$$

this is one to one and onto





Chapter Real numbers

Properties of addition and multiplication

I) $a+b=b+a$ $\qquad$ $a\cdot b = ba$

II) $(a+b)+c = a+(b+c)$ $\qquad$ $(ab)c = a(bc)$

III) $a(b+c) = ab+ac$

IV) $\exists\ 0\ \&\ 1 \in \mathbb{R}$, $0 \neq 1$, such that

$a+0 = a\ \&\ a\cdot 1 = a \quad \forall\ a \in \mathbb{R}$

V) $\forall a \in \mathbb{R} \quad \exists\ -a \in \mathbb{R}$ such that $a+-a=0$

$\forall a \in \mathbb{R}\ \&\ a \neq 0 \quad \exists\ a^{-1} \in \mathbb{R}$ such that $a . a^{-1} = 1$

Consequences:

1) $\forall a, b \in \mathbb{R}\ \exists!\ x \in \mathbb{R}$ such that

$x+a=b$

proof: $x = b+-a$ satisfies $x+a=b$

because $(b+-a)+a = b+(-a+a) = b+0 = b$ ✓

If $x+a=b \Rightarrow$, add $-a$ on both sides to get

$(x+a)+-a = b+-a$ $\qquad$ thus $x = b+-a$

$x + \overset{\shortparallel}{(a + -a)} = x + 0 = x$

Notation $b + -a = b - a$

Obs: $\forall a, b \in \mathbb{R}$ $a \neq 0$ $\exists! x$ such that $ax = b$.

Obs: $-(-a) = a$ Why? because $-(-a)$ is the unique number such that $-(-a) + -a = 0$.

but $a + -a = 0$ also thus, $a = -(-a)$

ORDER:

$\exists \mathbb{R}_+ \subset \mathbb{R}$ such that

(1) $a, b \in \mathbb{R}_+ \implies a+b$ and $ab \in \mathbb{R}_+$

(2) $a \in \mathbb{R}$, exactly one of the following is true $a \in \mathbb{R}_+$, $a = 0$ or $-a \in \mathbb{R}_+$.

$\mathbb{R}_+$ is called the set of positive numbers

If $-a \in \mathbb{R}_+$, we say that a is negative.

Notation $a > b$ means $a - b \in \mathbb{R}_+$

$b < a$ means the same

$a \geq b$ means $a > b$ or $a = b$

Obs: $a \in \mathbb{R}_+ \iff a > 0$

a negative $\iff \underset{\substack{\shortparallel \\ 0-a}}{-a} \in \mathbb{R}_+ \iff 0 > a$

$\forall a, b \in \mathbb{R}$ Exactly one of the following

$\boxed{01}$ $a, b \in \mathbb{R}$. Exactly one of the following is true

$a > b$ or $a = b$ or $a < b$

$\boxed{02}$ $a > b$ and $b > c \Rightarrow a > c$

pf: $a - c = \underbrace{\underset{\in \mathbb{R}_+}{(a-b)} + \underset{\in \mathbb{R}_+}{(b-c)}}_{\in \mathbb{R}_+} \in \mathbb{R}_+ \Rightarrow a > c$

$\boxed{04}$ $a > b > 0$ & $c \geq d > 0 \Rightarrow$

$a \cdot c > bd$

pf: $ac - bd$

we prove

$a - b > 0 \quad c > 0 \Rightarrow (a-b) \cdot c > 0$

$c - d \geq 0 \quad b > 0 \Rightarrow + (c-d) \cdot b \geq 0$

add

$(a-b) \cdot c + (c-d) \cdot b > 0$

distribute toget

$ac - bd > 0$

thus $ac > bd$.

Obs: $a < 0$ $b < 0$ then $ab > 0$

proof $-a > 0$ $-b > 0$. then $(-a)(-b) > 0$

$(-1)^2 ab > 0 \Rightarrow 1\, ab > 0 \Rightarrow ab > 0.$

$(-1)(-1) = -(-1) = 1$

Obs: $-a = (-1)a$

proof: $-1+1=0$

$((-1)+1)a = 0a$

$(-1)a + a = 0$

thus $(-1)a = -a$

$0a = 0$

$0 = 0+0$

$0a = (0+0)a$

$0a = 0a + 0a$

$(-0a) + 0a = \underbrace{(-0a) + (0a}+0a)$

$0 = 0 + 0a = 0a$

Similarly positive . negative = negative

negative + negative = negative

Obs: 1) $a^2 \geq 0$ and $a=0 \Leftrightarrow a=0$

2) $a>0$ so is $a^{-1}>0$

notation $\frac{b}{a} = b\,a^{-1}$ if $a \neq 0$

3) $a>b>0 \Rightarrow b^{-1} > a^{-1} > 0$

pf: $a-b \in \mathbb{R}_+$ $a^{-1} \in \mathbb{R}_+ \Rightarrow$

$(a-b)a^{-1} \in \mathbb{R}_+$ $b^{-1} \in \mathbb{R}_+ \Rightarrow$

$(a-b)a^{-1}b^{-1} \in \mathbb{R}_+$

$a a^{-1} b^{-1} - b a^{-1} b^{-1} \in \mathbb{R}_+$

$b^{-1} - a^{-1} \in \mathbb{R}_+$

$b^{-1} > a^{-1}$ ✓

Natural number

$1, \quad 2 = 1+1, \quad 3 = 1+2, \ldots$

$\mathbb{N} = \{1, 2, 3, \ldots\}$

Obs $\quad 1 > 0 \Rightarrow 2 > 1 \Rightarrow 3 > 2 \ldots$

$a > b \Rightarrow a+c > b+c$

$1 < 2 < 3 < 4 < \ldots$

$\mathbb{Z}$ integer.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\ldots < -2 < -1 < 0 < 1 < 2 < \ldots$

$\mathbb{Q} = \left\{ \frac{a}{b} : a \;\&\; b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$

Def: $n \in \mathbb{N}$ $a \in \mathbb{R}$, then $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$

If $a \neq 0$ $\quad a^0 = 1$

If $a \neq 0$ $\quad a^{-n} = (a^n)^{-1} = \frac{1}{a^n} \leftarrow$

Properties: 1) $a^n a^m = a^{n+m} \quad \forall n, m \in \mathbb{Z}$

Properties: 1) $a^n a^m = a^{n+m}$ $\quad \forall\, n, m \in \mathbb{Z}$

2) $(a^n)^m = a^{nm}$

3) $(ab)^n = a^n b^n$

Def: $a \in \mathbb{R}$. The absolute value of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Properties: 1) $|a| \geq 0$ and $|a| = 0 \iff a = 0$

2) $|ab| = |a||b|$ $\quad \forall\, a, b \in \mathbb{R}$

3) $|a|^2 = a^2$

4) $\boxed{|a+b| \leq |a| + |b|}$

proof: $-|a| \leq a \leq |a|$

$-|b| \leq b \leq |b|$

$-|a| - |b| \leq a+b \leq |a| + |b|$

$-(|a| + |b|) \leq a+b \leq |a| + |b|$

then $|a+b| \leq |a| + |b|$

because if $-s < x < s$ with $s \geq 0$ then $|x| \leq s$ (you do it)

Property: $\big||a| - |b|\big| \leq |a-b|$

proof: $a = a - b + b$ $\quad |\ a|$

proof: $a = a - b + b$

then $|a| \le |a-b| + |b|$

then $|a| - |b| \le |a-b|$

also $\underbrace{|b| - |a| \le |b-a|}$

and $\Rightarrow |b| - |a| \le |a-b|$

I have $x = |a| - |b|$ $s = |a-b|$

I have $x \le s$ & $-x \le s$.

then $|x| \le s$ then

$$||a| - |b|| \le |a-b|.$$

Obs: $|a_1 + a_2 + \dots + a_n| \le |a_1| + \underbrace{|a_2 + \dots + a_n|}_{\le\, |a_2| + |a_3 + \dots + a_n|}$

Continue to get

$$|a_1 + a_2 + \dots + a_n| \le |a_1| + |a_2| + \dots + |a_n|$$

Invertals

Let $a, b \in \mathbb{R}$

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$

$[a,b) = \{x \in \mathbb{R} : a \le x < b\}$

$(a,b] = \{x \in \mathbb{R} : a < x \le b\}$

$$[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$$

$$(a,\infty) = \{x \in \mathbb{R} : a < x\}$$

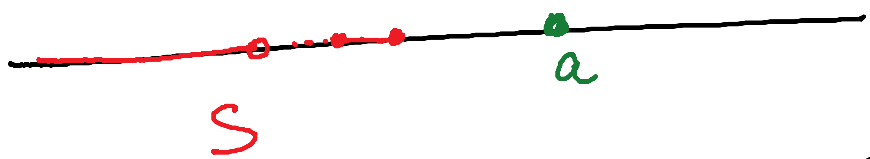
$$(-\infty,b] = \{x \in \mathbb{R} : x \le b\}$$

<u>Obs</u>: $|x-a| < \varepsilon \iff -\varepsilon < x-a < \varepsilon$

$\iff -\varepsilon + a < x < a + \varepsilon$

$\iff x \in (a-\varepsilon, a+\varepsilon)$

$\boxed{2.3}$ <u>Def</u>: 1) $S \subset \mathbb{R}$. An upper bound for the set S is a number a such that $s \le a$ $\forall s \in S$.



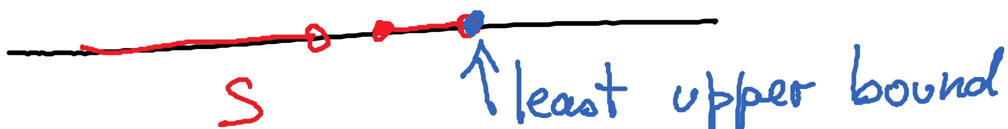
<u>Example</u> 1) $S = [0,2]$ $\quad a = 3$ is an upper bound for S.

<u>Def</u>: If S has an upper bound, we say that S is bounded from above.

<u>Def</u>: $S \subseteq \mathbb{R}$. y is a least upper bound if

1) y is an upper bound.

2) If b is an upper bound of S, $\Rightarrow y \leq b$.



<u>Obs</u>: least upper bounds are unique

<u>proof</u>: Let $S \subseteq \mathbb{R}$. Let y_1 & y_2 be two least upper bounds of S.

Notation $\quad lub$ = least upper bound
$\qquad\qquad\; ub$ = upper bound

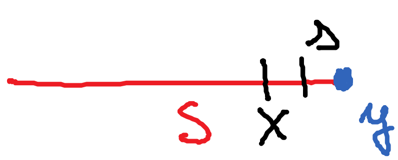
$y_1 \leq y_2$ because y_1 is a lub & y_2 is an ub

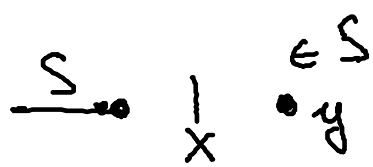
$y_2 \leq y_1$ because y_2 is a lub & y_1 is an ub

then $y_1 = y_2$

<u>Notation</u>: $lub(S)$

<u>Obs</u>: Let $y = lub(S)$. Let $x < y$. then there exist $s \in S$ such that $x < s \leq y$





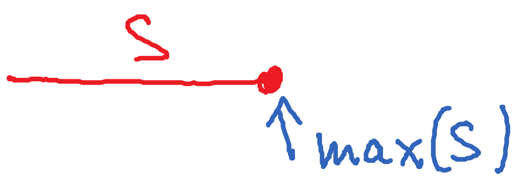
proof: Since $y = lub(S)$ & $x < y \Rightarrow x$ is not an upper bound of S. Then $\exists\, s \in S$ such that $s > x$. But y is an upper bound of $S \Rightarrow s \leq y$.

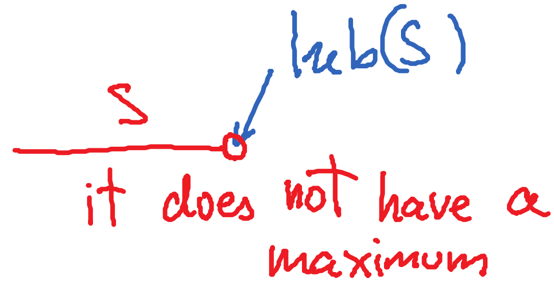
<u>Def</u>: $S \subset \mathbb{R}$

$\max(S)$ = maximum of S.

$\max(S)$ is a number that satifies

1) it is an upper bound of S
2) it belongs to S





<u>Obs</u>: $\max(S)$ need not exist

<u>Obs</u>: If $\max(S)$ exists $\Rightarrow$ $\max(S) = \text{lub}(S)$, and thus, it is unique.

<u>Example</u> $S = (0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$.

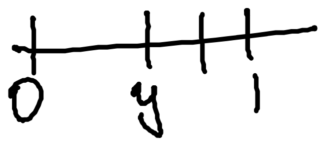
$\text{lub}(S) = 1$

<u>proof</u>: if $x \in S \Rightarrow x < 1 \Rightarrow 1$ is an upper bound of S.

Let $y < 1$ if $y \leq 0 \Rightarrow y < \frac{1}{2} \in S \Rightarrow$ y is not an upper bound of S

If $0 < y < 1 \Rightarrow 0 < y < \frac{y+1}{2} < 1 \Rightarrow$

$\frac{y+1}{2} \in S \Rightarrow y$ is not an



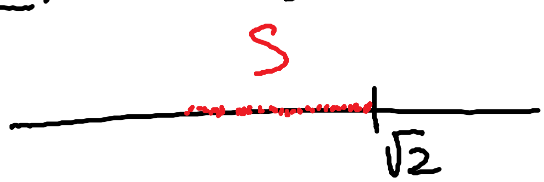
$\frac{y+1}{2} \in S \Rightarrow y$ is not an upper bound of S.

Then $1 = \text{lub}(0,1)$

Obs: If S is finite $\Rightarrow \max(S)$ exists.

Property: If $S \neq \emptyset$ & S is bounded from above $\Rightarrow$ S has a lub

Ex: $S = \{x \in \mathbb{Q} : x < \sqrt{2}\}$



$\text{lub}(S) = \sqrt{2}$

Def: lower bound ......

bounded from below.....

greatest lower bound glb

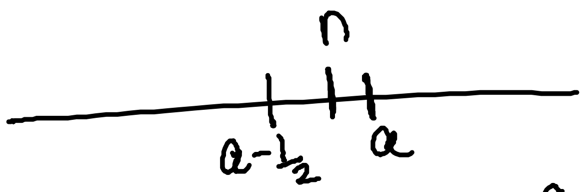
Proposition: $\forall x \in \mathbb{R}$ $\exists n \in \mathbb{N}$ such that $n > x$

proof: Assume this is not true.

Thus, $\exists x \in \mathbb{R}$ such that $n \leq x$ $\forall n \in \mathbb{N}$.

then $\mathbb{N}$ is bounded from above $\Rightarrow$ $\mathbb{N}$ has a least upper bound. Let $a = \text{lub}(\mathbb{N})$

then $a - \frac{1}{2} < a$



$\exists\, n \in \mathbb{N}$ such that

$a - \frac{1}{2} < n \leq a$ add 1

To get $a < a + \frac{1}{2} < n+1 \in \mathbb{N}$ contradiction

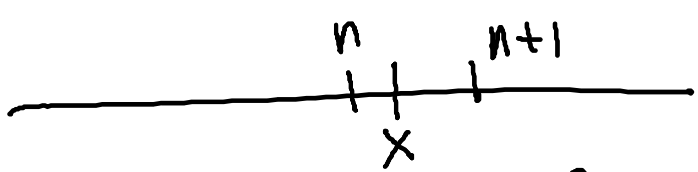
thus $\mathbb{N}$ is not bounded from above

<u>Obs</u>: $\forall\, \varepsilon > 0$ $\exists\, n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$

<u>proof</u>: Since $\mathbb{N}$ is not bounded, $\exists\, n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$ $\Rightarrow$ $\varepsilon > \frac{1}{n}$ ✓

<u>Obs</u>: $\forall\, x \in \mathbb{R}$ $\exists\, n \in \mathbb{Z}$ such that

$n \leq x < n+1$

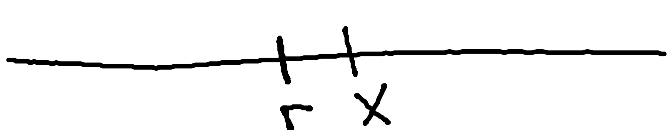


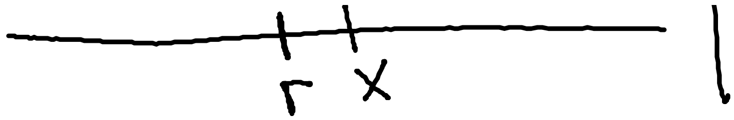
<u>proof</u>:

$S = \{ k \in \mathbb{Z} : k \leq x \}$

Let $r = \text{lub}(S)$ We need to show that $r \in \mathbb{Z}$ and $r \leq x < r+1$

$\boxed{r \leq x}$ because $r = \text{lub}(S)$ & x is an ub of S

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