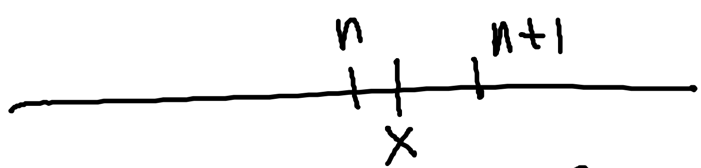
Obs: $\forall x \in \mathbb{R}\ \exists n \in \mathbb{Z}$ such that $n \leq x < n+1$

proof:



$S = \{k \in \mathbb{Z} : k \leq x\}$

Let $r = \mathrm{lub}(S)$. We need to show that $r \in \mathbb{Z}$ and $r \leq x < r+1$

$\boxed{r \leq x}$ because $r = \mathrm{lub}(S)$ & x is an ub of S

$r - \frac{1}{2}$ is not an upper bound of S because r is the least upper bound of S. Thus, $\exists k \in S$ such that $r - \frac{1}{2} < k \leq r$.

If $t \in \mathbb{Z}$ & $t > k \Rightarrow t \geq k+1 > r + \frac{1}{2} \Rightarrow t \notin S$

If $t \in S \Rightarrow t \leq k \Rightarrow k$ is an upperbound of $S \Rightarrow k = r = \mathrm{lub}(S)$.

$r+1 \notin S \Rightarrow r+1 > x$.

Lemma: $\forall x \in \mathbb{R}$ & $\forall m \in \mathbb{N}\ \exists n \in \mathbb{Z}$ such that $\frac{n}{m} \leq x < \frac{n+1}{m}$

such that $\frac{n}{m} \le x < \frac{n+1}{m}$

proof: Apply last Obs to mx. We can find $n \in \mathbb{Z}$ such that

$$n \le mx < n+1$$

Devide by m $\quad \frac{n}{m} \le x < \frac{n+1}{m}$

Prop: $\forall x \in \mathbb{R}$ $\forall \varepsilon > 0$ $\exists r \in \mathbb{Q}$ such that $|x - r| < \varepsilon$.

proof: Select m such that $m > \frac{1}{\varepsilon}$ & $m \in \mathbb{N}$. From last lemma $\exists n \in \mathbb{Z}$ such that $\frac{n}{m} \le x < \frac{n+1}{m} \implies$

$$\left|x - \frac{n}{m}\right| < \frac{1}{m} < \varepsilon \qquad r = \frac{n}{m}$$

Prop: Let $a \in \mathbb{R}$ & $a > 0$. Then, $\exists!\, b > 0$ such that $b^2 = a$

proof: Assume $0 < b_1 < b_2$ then $b_1^2 < b_2^2$. This proves uniqueness.

Let $S = \{x \in \mathbb{R} : x > 0 \text{ \& } x^2 \le a\}$

$(1+a)^2 = 1+2a+a^2 > a$

thus $1+a$ is a bound of S. S is bounded from above.

Let $b = \text{lub}(S)$. I want to show that $b^2 = a$

Let $\varepsilon > 0$ & $\varepsilon < b$ then

$$(b-\varepsilon)^2 < b^2 < (b+\varepsilon)^2$$

$b+\varepsilon \notin S$ because b is an upper bound of S and $b+\varepsilon > b$. thus, $(b+\varepsilon)^2 > a$.

$(b-\varepsilon)^2 < a$ because, if $(b-\varepsilon)^2 \geq a \Rightarrow$ $b-\varepsilon$ would be an upper bound of S but $b = \text{lub}(S)$ & $b-\varepsilon < b$ (impossible).

then, $\forall \varepsilon > 0$

$$\begin{array}{rl} & (b-\varepsilon)^2 < a < (b+\varepsilon)^2 \\ + & -(b+\varepsilon)^2 < -b^2 < -(b-\varepsilon)^2 \\ \hline & -4b\varepsilon < a-b^2 < 4b\varepsilon \end{array}$$

$$|a-b^2| < 4b\varepsilon \qquad \forall\ \varepsilon > 0$$

~~take~~ If $|a-b^2| \neq 0$

select $\varepsilon = \dfrac{|a-b^2|}{4b}$ to get $|a-b^2| < |a-b^2|$

select $\varepsilon = \dfrac{|a-b^2|}{4b}$ to get ...

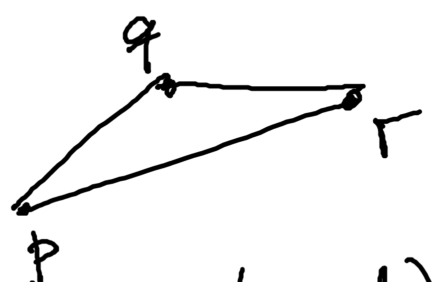
impossible.

<u>notation</u>: $b = \sqrt{a}$

<u>Metric Spaces</u>

<u>Def</u>: A metric space is a set E together with $d : E \times E \to \mathbb{R}$ such that:

1) $d(p,q) \geq 0 \quad \forall\, p,q \in E$ & $d(p,q) = 0$ if and only if $p = q$
2) $d(p,q) = d(q,p) \quad \forall\, p,q \in E$
3) $d(p,r) \leq d(p,q) + d(q,r) \quad \forall\, p,q,r \in E$



<u>Notation</u> (E, d) $\quad d(p,q)$ = distance from p to q

<u>Example</u>: 1) $E = \mathbb{R}$ $\quad d(p,q) = |p - q|$

<u>proof</u>: 1) ✓ 2) ✓ 3)

$$p - r = (p-q) + (q-r)$$

$$|p-r| \leq |p-q| + |q-r| \quad ✓$$

$|p-r| \le |p-q| + |q-r|$ ✓

$d(p,r) \le d(p,q) + d(q,r)$

Ex: $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{R} \ 1 \le i \le n\}$

$x = (x_1, x_2, \dots, x_n) \quad y = (y_1, y_2, \dots, y_n)$

$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2}$

Prop: (Schwartz inequality)

$a_i, b_i \in \mathbb{R} \quad 1 \le i \le n$. then

$$\left|\sum_{i=1}^n a_i b_i\right| \le \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}$$

proof: $\forall \ \alpha, \beta \in \mathbb{R}$ we have

$$0 \le \sum_{i=1}^n (\alpha a_i - \beta b_i)^2 = \alpha^2 \sum_{i=1}^n a_i^2 - 2\alpha\beta \sum_{i=1}^n a_i b_i +$$

$$+ \beta^2 \sum_{i=1}^n b_i^2$$

Set $\alpha = \sqrt{\sum_{i=1}^n b_i^2}$. $\beta = \pm\sqrt{\sum_{i=1}^n a_i^2}$

$$0 \le \left(\sum_{i=1}^n b_i^2\right)\left(\sum_{i=1}^n a_i^2\right) \mp 2\sqrt{\sum_{i=1}^n b_i^2}\sqrt{\sum_{i=1}^n a_i^2} \sum_{i=1}^n a_i b_i$$

$$+ \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$$

$+ \left(\sum_{i=1} \ldots\right)\left(\sum_{i=1} \ldots\right)$

Devide by $2\sqrt{\sum_{i=1}^n b_i^2}\sqrt{\sum_{i=1}^n a_i^2}$

$$\pm \sum_{i=1}^n a_i b_i \le \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}$$

$$\left|\sum_{i=1}^n a_i b_i\right| \le \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}$$

Corollary $\sqrt{\sum_{i=1}^n (a_i+b_i)^2} \le \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}$

proof: $\sum_{i=1}^n (a_i+b_i)^2 = \sum_{i=1}^n a_i^2 + 2\sum_{i=1}^n a_i b_i + \sum_{i=1}^n b_i^2 \le \sum_{i=1}^n a_i^2 + 2\sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2} + \sum_{i=1}^n b_i^2 =$

$$= \left(\sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}\right)^2$$

Prop: $d(x,y) = \sqrt{\sum_{i=1}^n (x_i-y_i)^2}$ is a distance in $\mathbb{R}^n$

proof: $a_i = x_i - z_i$ $\quad b_i = z_i - y_i$

$$d(x,y) = \sqrt{\sum_{i=1}^n (x_i-y_i)^2} \le \sqrt{\sum_{i=1}^n (x_i-z_i)^2} + \sqrt{\sum_{i=1}^n (z_i-y_i)^2}$$

$= d(x,z) + d(z,y)$

Examples of metric spaces

1) (E, d) is a metric space and $E' \subset E$ then (E', d) is a metric space

2) E any set $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

Prop: $x_1, \ldots, x_n \in E$. (E, d) metric space, then

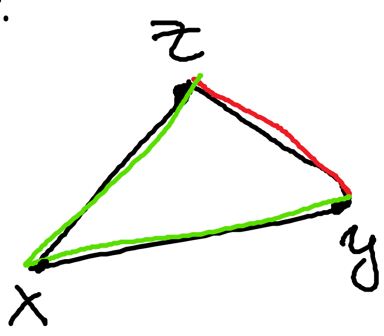
$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$$

proof: (Practice)

Prop: $x, y, z \in E$, then

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

proof:



$d(x,z) \leq d(x,y) + d(y,z)$

$d(x,z) - d(y,z) \leq d(x,y)$

reverse role of x & y

$d(y,z) - d(x,z) \leq d(y,x)$

then $|d(x,z) - d(y,z)| \leq d(x,y)$

Open and closed set

Def: (E, d) metric space. $x_0 \in E$, $r > 0$. The open ball of radius r centered at x_0 is

$$B_r(x_0) = \{ x \in E : d(x, x_0) < r \}$$

The closed ball of radius r centered at x_0 is

$$C_r(x_0) = \{ x \in E : d(x, x_0) \leq r \}$$

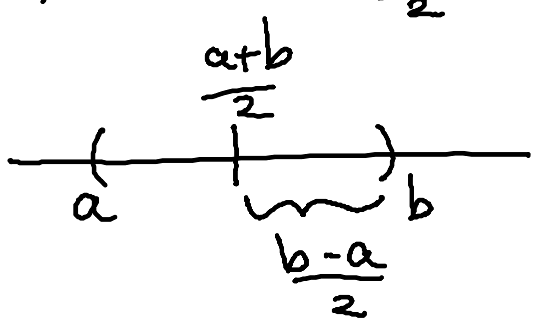
Recall: If $a, b \in \mathbb{R}$, then

$(a,b) = \{ x \in \mathbb{R} : a < x < b \}$ open interval

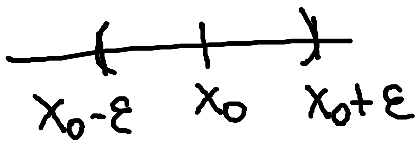
$[a,b] = \{ x \in \mathbb{R} : a \leq x \leq b \}$ closed interval

Examples

1) $(a,b) = B_{\frac{b-a}{2}}\left(\frac{b+a}{2}\right)$

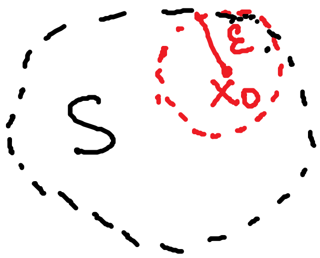


2)

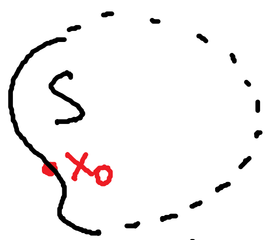


$(x_0 - \varepsilon, x_0 + \varepsilon) = B_\varepsilon(x_0)$

Def: $S \subset E$. S is open if $\forall x \in S$ $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset S$.



open



not open

Prop: 1) $\emptyset$ is open
2) E is open
3) Union of open sets is open

proof: 3) A_α open for all $\alpha \in I$. Let $A = \bigcup_{\alpha \in I} A_\alpha$

Let $x \in A$, I need to show that $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset A$. Since $x \in A = \bigcup_{\alpha \in I} A_\alpha$, $\exists \alpha_0 \in I$ such that $x \in A_{\alpha_0}$. Since A_{α_0} is open and $x \in A_{\alpha_0}$ $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset A_{\alpha_0}$. Since $A_{\alpha_0} \subset A$, then $B_\varepsilon(x) \subset A$ ✓

Prop: $A, B \subset E$ both open, then $A \cap B$ is open.

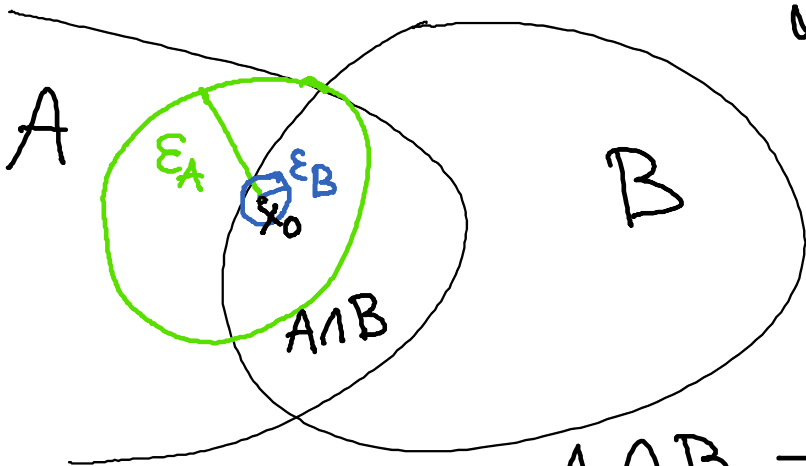
proof: Let $x_0 \in A \cap B$. I need to show
$B_\varepsilon(x_0) \subset A \cap B$

Proof: Let $x_0 \in A \cap B$. ... $B_\varepsilon(x_0) \subset A \cap B$

that $\exists\ \varepsilon > 0$ such that

Since A is open $\exists\ \varepsilon_A > 0$ such that $B_{\varepsilon_A}(x_0) \subset A$

Since B is open $\exists\ \varepsilon_B > 0$ such that $B_{\varepsilon_B}(x) \subset B$



Let $\varepsilon = \min\{\varepsilon_A, \varepsilon_B\}$.

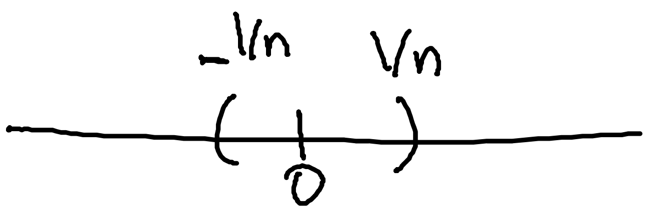
then

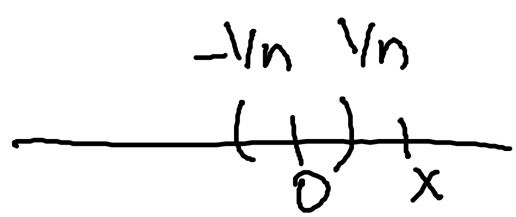
$B_\varepsilon(x_0) \subset B_{\varepsilon_A}(x_0) \subset A$

$B_\varepsilon(x_0) \subset B_{\varepsilon_B}(x_0) \subset B$

then $B_\varepsilon(x_0) \subset A \cap B \Rightarrow A \cap B$ is open.

Example: $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$





Each $\left(-\frac{1}{n}, \frac{1}{n}\right)$ is open but $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$

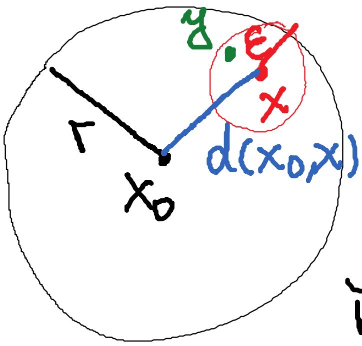
which is not open because

$0 \in \{0\}$ but $\forall\ \varepsilon > 0$, $\frac{\varepsilon}{2} \in B_\varepsilon(0)$ ...

but $\frac{\varepsilon}{2} \notin \{0\} \Rightarrow B_\varepsilon(0) \not\subset \{0\} \quad (\forall \varepsilon > 0)$

Prop: Open balls are open

proof: Let $x_0 \in E$. Let $r > 0$. We want to show that $B_r(x_0)$ is open.



Let $x \in B_r(x_0)$ I w s that $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset B_r(x_0)$.

Let $\varepsilon = r - d(x_0, x)$

Since $x \in B_r(x_0)$, $d(x, x_0) < r \Rightarrow \varepsilon > 0$.

Let $y \in B_\varepsilon(x) \Rightarrow d(x, y) < \varepsilon = r - d(x_0, x)$

Ins that $y \in B_r(x_0)$ or $d(y, x_0) < r$

$$d(x_0, y) \leq d(x_0, x) + \underbrace{d(x, y)}_{< r - d(x_0, x)} < r \quad \checkmark$$

Ex: E a set $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

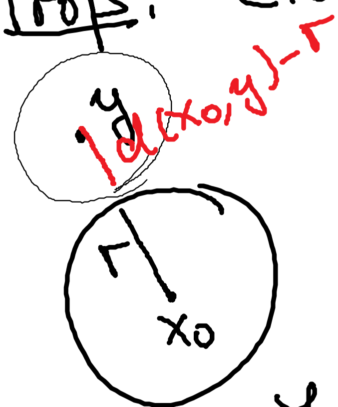
$B_1(x) = \{x\}$. Any set in E with the discrete distance is open.

$$S \subset E \qquad S = \bigcup_{x \in S} \underbrace{\{x\}}_{\text{open}}$$

Def: S is closed if S^c is open

closed open neither.

Prop: Closed balls are closed



proof: Let $x_0 \in E$ & $r > 0$.

$C_r(x_0) = \{ x \in E : d(x, x_0) \leq r \}$.

Nts that $(C_r(x_0))^c$ is open.

Let $y \in (C_r(x_0))^c$. Then $d(y, x_0) > r$

Let $\varepsilon = d(y, x_0) - r$

Let $z \in B_\varepsilon(y)$

$$d(y, x_0) \leq d(y, z) + d(z, x_0)$$

$$d(z, x_0) \geq d(y, x_0) - d(y, z) > d(y, x_0) - \varepsilon =$$

$$= d(y, x_0) - (d(y, x_0) - r) = r$$

$$d(z, x_0) > r \quad \checkmark$$