Proposition: $x_n \in \mathbb{R}$, x_n is bounded. Then, there exists a subsequence of x_n that is decreasing or increasing.

Proof: Let $a_n = \text{lub}\{x_n, x_{n+1}, \cdots\}$

$\Rightarrow a_n \geq a_{n+1} \geq a_{n+2} \cdots$,

$\Rightarrow a_n$ is an upper bound of $\{x_{n+1}, x_{n+2}, \cdots\}$

but a_{n+1} is the lub of $\{x_{n+1}, x_{n+2}, \cdots\}$.

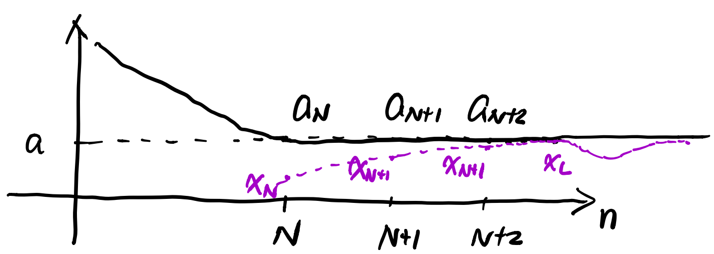
$\Rightarrow a_{n+1} \leq a_n$. So a_n is a decreasing sequence.

Since x_n is bounded (from below), $\exists\, C \in \mathbb{R}$, s.t. $x_n \geq C$, $\forall n$.

But, $a_n \geq x_n \Rightarrow a_n \geq C$, $\forall n$

Thus, $a_n \to a = \text{glb}\{a_1, a_2, \cdots\}$.

Case 1, $\exists\, N \in \mathbb{N}$, s.t. $a_N = a$.



since $a = a_N \geq a_n \geq a$ $\forall n \geq N$, $a_n = a$ $\forall n \geq N$.

Let $I = \{ n \in \mathbb{N} : x_n = a \}$.

① If $\#(I) = \infty$, then let $n_1 < n_2 < \cdots$, $n_i \in I$.

So, $x_{n_1} = x_{n_2} = \cdots = x_{n_r} = \cdots = a$ is a decreasing subsequence of x_n.

② If $\#(I) < \infty$, then

Let $M = \max\limits_{n \in I} n$, then $x_n \neq a \quad \forall n > M$.

Let $L = \max\{N, M+1\}$. then if $n \geq L$, $a_n = a$ and $x_n < a$.

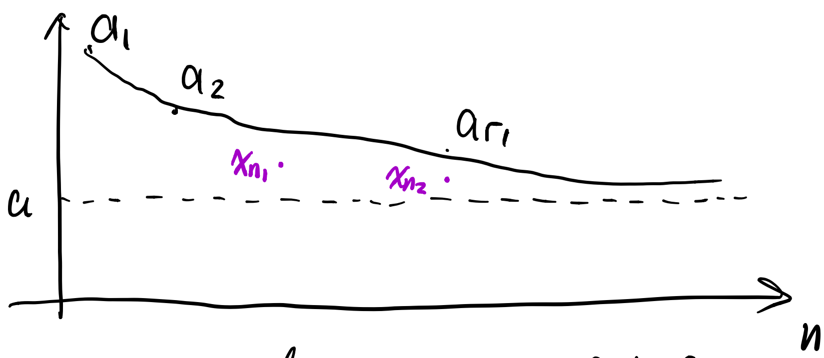
Let $n_1 = L$, $\varepsilon_1 = a_{n_1} - x_{n_1} > 0$, $a_{n_1} = a$

so, we can select $n_2 > n_1$, s.t $x_{n_1} < x_{n_2}$

because $a_{n_1} = \text{lub}\{x_n, x_{n+1}, \cdots\}$.

We can keep doing it for x_{n_i} such that x_{n_i} is an increasing subsequence of x_n

Case 2: $a_n > a$, $\forall n \in \mathbb{N}$.



$a_1 > a$, but $a_1 = \text{lub}\{x_1, x_2, \cdots\}$.

Then, $\exists\, n_1$, s.t. $x_{n_1} > a$

Since $a_n \to a$ and decreasing, $\exists\, r_1 > n_1$ s.t.

$x_{n_1} > a_{r_1} \geq x_{r_1}$.

Select $n_2 \geq r_1 > n_1$, s.t. $a < x_{n_2} \leq a_{r_1} < x_{n_1}$

We can keep doing that and have $n_3 > n_2$

s.t. $a < x_{n_3} \leq a_{r_2} < x_{n_2}$.

So, such x_{n_i} is a subsequence of x_n that is decreasing to a.

<u>Corollary</u>: If $x_n \in \mathbb{R}$, x_n is bounded, then $\exists\, x_{n_i}$ converges.

<u>Proof</u>. By the proposition, $\exists\, x_{n_i}$ that is

increasing or decreasing. But since it is also bounded, x_{n_i} converges.

<u>Lemma</u> $S \subset \mathbb{R}$, S is bounded and closed. $S \subset \bigcup_{i \in I} U_i$ U_i open, $\forall i \in I$. Then $\exists\, \delta > 0$ s.t. $\forall x \in S$, $\exists\, i_x \in I$ s.t. $(x-\delta, x+\delta) \subset U_{i_x}$.

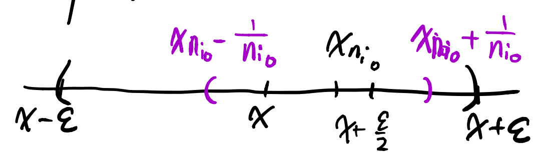
~~proof~~: If the lemma is not true, $\exists\, x_n \in S$, s.t. $(x_n - \frac{1}{n}, x_n + \frac{1}{n})$ is not included in one of the open sets U_i.

Since S is bounded, $x_n \in S$, <u>x_n is bounded</u>. Then, $\exists\, x_{n_i} \to x$

Since S is closed and $x_{n_i} \in S$, $x \in S$.

Since $x \in S \subset \bigcup_{i \in I} U_i$, $\exists\, i_0$ s.t. $x \in U_{i_0}$

Since U_{i_0} is open, $\exists\, \varepsilon > 0$, s.t. $(x-\varepsilon, x+\varepsilon) \subset U_{i_0}$.



Since $x_{n_i} \to x$, $\exists\ n_{i_0}$ s.t. $|x_{n_{i_0}} - x| < \frac{\varepsilon}{2}$.

and $\frac{1}{n_{i_0}} < \frac{\varepsilon}{2}$.

Then, $(x_{n_{i_0}} - \frac{1}{n_{i_0}}, x_{n_{i_0}} + \frac{1}{n_{i_0}}) \subset (x-\varepsilon, x+\varepsilon) \subset U_{i_0}$.

This contradicts with that $(x_n - \frac{1}{n}, x + \frac{1}{n})$ is not included in one of the U_i $\forall i \in I$.

<u>Theorem</u> If $S \subset \mathbb{R}$, S is closed and bounded. Then, S is compact.

<u>Proof</u> Let $\bigcup_{i \in I} U_i \supset S$ where U_i is open $\forall i \in I$.

From last lemma, $\exists\ \delta > 0$, s.t. $\forall x \in S$, $\exists$ $i_x \in I$ s.t. $(x-\delta, x+\delta) \subset U_{i_x}$

Let $x_1 = \min S = \text{glb}(S)$ since S is closed

and $S_1 = S - (x_1 - \delta, x_1 + \delta)$ is closed.

Let $x_2 = \min S_1 = \text{glb}(S_1)$ since S_1 is closed.

and $S_2 = S_1 - (x_2 - \delta, x_2 + \delta)$ is closed.

Thus, $x_n - x_1 \geqslant (n-1)\delta$ if $S_n \neq \phi$.

Since S is bound, S_n is ϕ for some n.

Thus, $S \subset \bigcup_{j=1}^{n-1} (x_j - \delta, x_j + \delta)$

since each $(x_j - \delta, x_j + \delta) \subset U_{i_{x_j}}$

$$S \subset \bigcup_{j=1}^{n-1} (x_j - \delta, x_j + \delta) \subset \bigcup_{j=1}^{n-1} U_{i_{x_j}} .$$