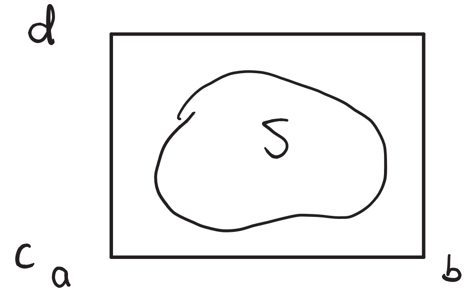
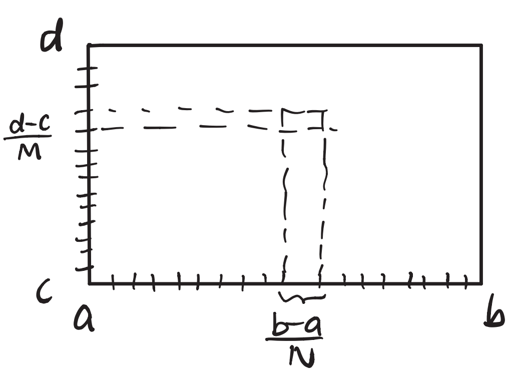
Theorem $S \subset \mathbb{R}^2$, S closed and bounded $\Rightarrow$ S is compact.

proof: S is bounded, $\exists [a,b] \times [c,d] \supset S$ for some $a, b, c, d \in \mathbb{R}$.



If we prove that $[a,b] \times [c,d]$ is compact, then its closed subset S is also compact.
(closed subset of a compact set is also compact).
So, we can assume $S = [a,b] \times [c,d]$.

Let $S \subset \bigcup_{i \in I} U_i$ for each U_i is open.
Claim 1: $\exists\ \delta > 0$ s.t. $\forall x \in S$, $\exists\ i_x$ s.t. $B_\delta(x) \subset U_{i_x}$. (prove it later) (Lemma)



$$\Delta x = \frac{b-a}{N} < \frac{\delta}{\sqrt{2}}$$

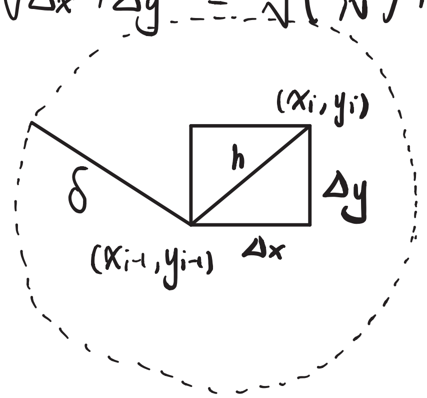
$$\Delta y = \frac{d-c}{M} < \frac{\delta}{\sqrt{2}}$$

Select $N > \frac{\sqrt{2}(b-a)}{\delta}$, $M > \frac{\sqrt{2}(d-c)}{\delta}$

$$(x_i, y_i) = (a + i\Delta x, \; c + j\Delta y)$$

So, $S = \bigcup_{i=1}^{N} \bigcup_{j=1}^{M} [x_{i-1}, x_i] \times [y_{j-1}, y_j]$

$$h = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\left(\frac{b-a}{N}\right)^2 + \left(\frac{d-c}{M}\right)^2} < \sqrt{\left(\frac{\delta}{\sqrt{2}}\right)^2 + \left(\frac{\delta}{\sqrt{2}}\right)^2} = \delta .$$



so, $[x_{i-1}, x_i] \times [y_{i-1}, y_i] \subset B_\delta(x_{i-1}, y_{i-1})$

$$\subset U_{i_{(x_{i-1}, y_{i-1})}}$$

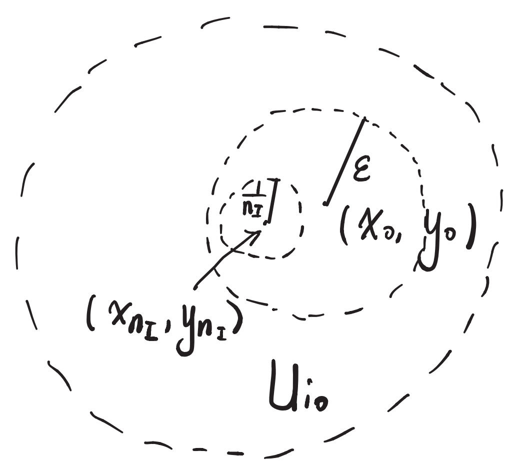
then $S \subset \bigcup_{i=1}^{N} \bigcup_{j=1}^{M} U_{i_{(x_{i-1}, y_{i-1})}}$.

<u>Lemma</u> (claim 1) $[a,b]\times[c,d] \subset \bigcup_{i\in I} U_i$, then $\exists\, \delta>0$, $\forall x \in S$ $\exists\, i_x$ s.t. $B_\delta(x) \subset U_{i_x}$

~~proof~~: By contradiction. If this is not true, $\exists\, (x_n, y_n) \in S$, s.t. $B_{\frac{1}{n}}(x_n, y_n) \not\subset U_i$, $\forall i \in I$.

<u>Claim 2</u>: $\exists\, n_i$ s.t. $(x_{n_i}, y_{n_i}) \to (x, y)$

$\exists\, i_0$ such that $(x_0, y_0) \in U_{i_0}$ open. Then $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x_0, y_0) \subset U_{i_0}$



$\Rightarrow \exists\, I$ such that $\frac{1}{n_I} < \frac{\varepsilon}{2}$, and $d((x_{n_I}, y_{n_I}), (x_0, y_0)) < \frac{\varepsilon}{2}$

$\Rightarrow B_{\frac{1}{n_I}}(x_{n_I}, y_{n_I}) \subset B_\varepsilon(x_0, y_0) \subset U_{i_0}$

This is a contradiction.

Lemma (Claim 2): (x_n, y_n) bounded in $\mathbb{R}^2$
$\Rightarrow \exists\ n_i$ s.t. (x_{n_i}, y_{n_i}) converges.

proof: Since (x_n, y_n) bounded in $\mathbb{R}^2$,
x_n and y_n are bounded in $\mathbb{R}$.
$\Rightarrow \exists\ x_0$ and x_{n_i} such that $x_{n_i} \to x_0$.
$\Rightarrow \exists\ y_0$ and $y_{n_{i_k}}$ such that $y_{n_{i_k}} \to y_0$
$\Rightarrow$ since $x_{n_{i_k}}$ is a subsequence of x_{n_i}, $x_{n_{i_k}} \to x_0$
$\Rightarrow$ (x_n, y_n)'s subsequence $(x_{n_{i_k}}, y_{n_{i_k}}) \to (x_0, y_0)$.

Connectedness

Def: E is connected if and only if E and $\emptyset$ are the only sets that are both open and closed.

Example $\mathbb{Q}$ is not connected. Since
$S = \{x \in \mathbb{Q} : x < \sqrt{2}\}$ is both open and closed

proof: ① S is open. Let $x \in S$, $\varepsilon = \sqrt{2} - x > 0$, then $(x-\varepsilon, x+\varepsilon) = (x-\varepsilon, \sqrt{2}) \subset S$

② S is closed, because S^c is open

Let $x \in S^c$, $\varepsilon = x - \sqrt{2} > 0$ ($\sqrt{2} \notin \mathbb{Q}$), then

$(x-\varepsilon, x+\varepsilon) = (\sqrt{2}, x+\varepsilon) \subset S^c$

Thus, S is both open and closed.

<u>Lemma</u>: Let $S \subset E$, A is open in S iff $A = B \cap S$ where B is open in E.

<u>Example</u> $S = [0,1]$, $A = [0,1)$ is open in S.

proof: Let $x \in A$, $\varepsilon = 1 - x$,

then $B_\varepsilon(x) = \{x \in S ; |x - x_0| < \varepsilon\}$

$= (x_0 - \varepsilon, x+\varepsilon) \cap S$

$= (x_0 - \varepsilon, 1) \cap S$

$= [0, 1) \subset A$.

$A = (-\infty, 1) \cap S$, $B = (-\infty, 1)$ is open in $\mathbb{R}$.

Proof of the Lemma:

$\Rightarrow$) Let $x \in S$, $B_\varepsilon^E(x) = \{y \in E : d(x,y) < \varepsilon\}$.

$B_\varepsilon^S(x) = \{y \in S : d(x,y) < \varepsilon\}$.

So, $B_\varepsilon^S(x) = B_\varepsilon^E(x) \cap S \quad (S \subset E)$.

If A is open in S, $\forall x \in A$, $\exists \varepsilon_x > 0$ s.t. $B_{\varepsilon_x}^S(x) \subset A$. Then $A = \bigcup_{x \in A} B_{\varepsilon_x}^S(x) = \bigcup_{x \in A} \left(B_{\varepsilon_x}^E(x) \cap S\right)$

$$= \left(\bigcup_{x \in A} B_\varepsilon^E(x)\right) \cap S.$$

Then, let $B = \bigcup_{x \in A} B_\varepsilon^E(x)$, B is open in E.

$\Leftarrow$) Let $x \in A = B \cap S$, we need to show that $\exists \varepsilon > 0$ s.t. $B_\varepsilon^S(x) \subset A$.

Since B is open in E, $\exists \varepsilon > 0$, s.t

$B_\varepsilon^E(x) \subset B \Rightarrow B_\varepsilon^S(x) = S \cap B_\varepsilon^E(x) \subset S$.