Note Title 5/11/2015

Math 6701

people.math.gatech.edu/~ggold

<u>Vectors</u>

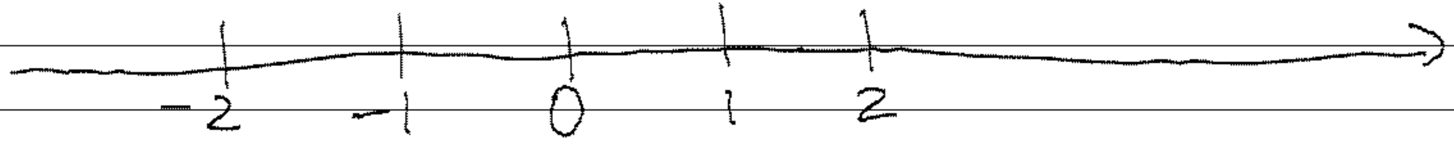
$\mathbb{R}$ = set of real numbers

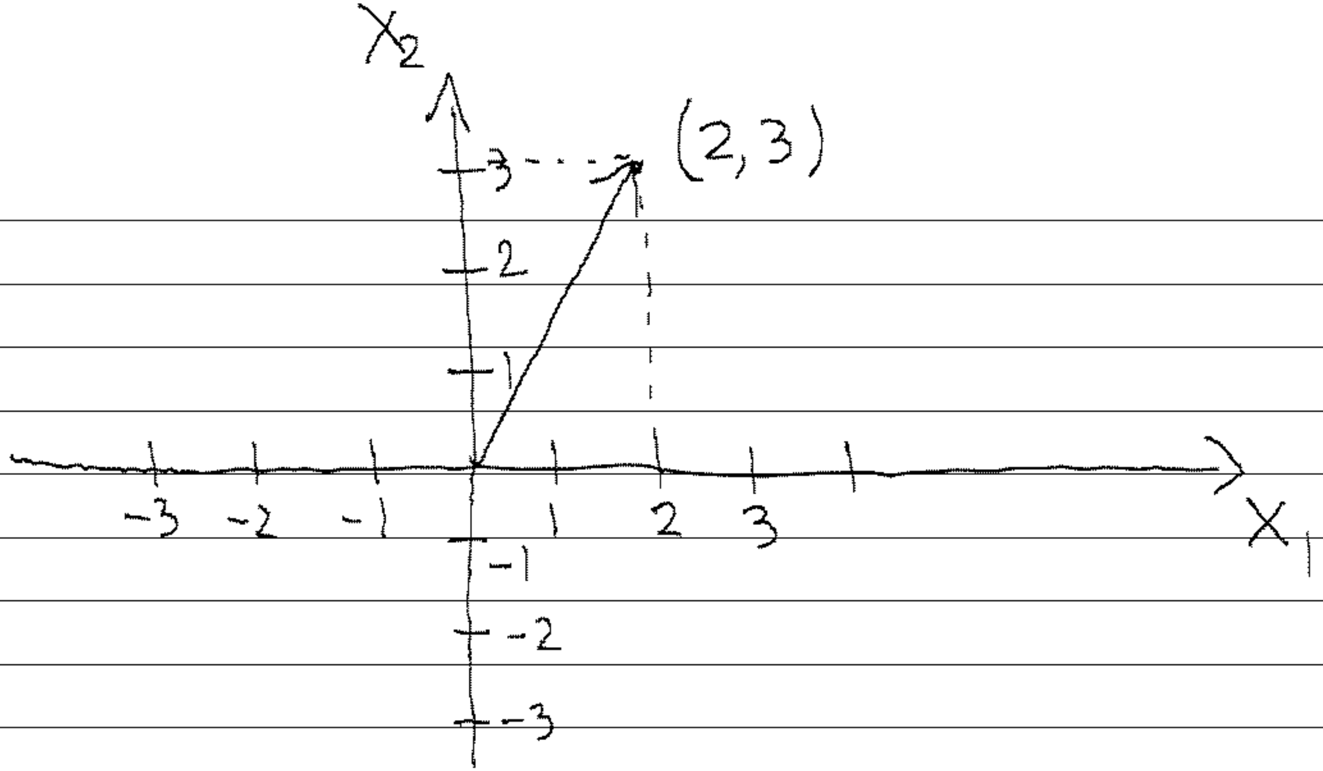
$\mathbb{R}^2 = \{ (x,y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R} \}$ Elements in $\mathbb{R}^n$ are called points or vectors

Example: $(2,1) \in \mathbb{R}^2$

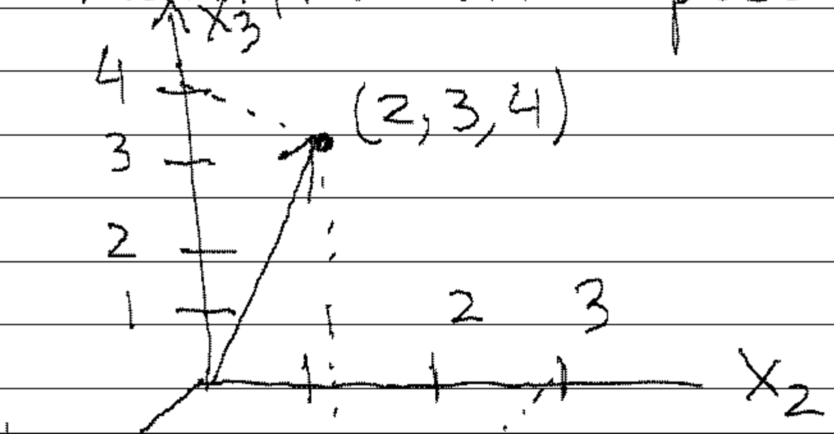
$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R} \}$

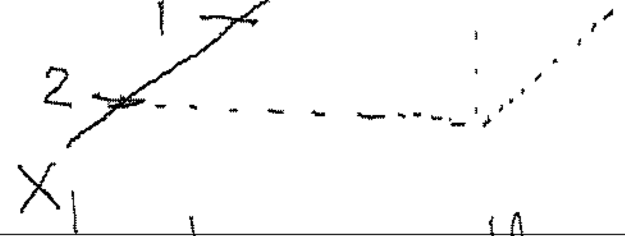
<u>Geometric interpretation</u> Real line





$\mathbb{R}^3$ is identified with space





<u>Operations with vectors</u>

1) Vector-scalar multiplication

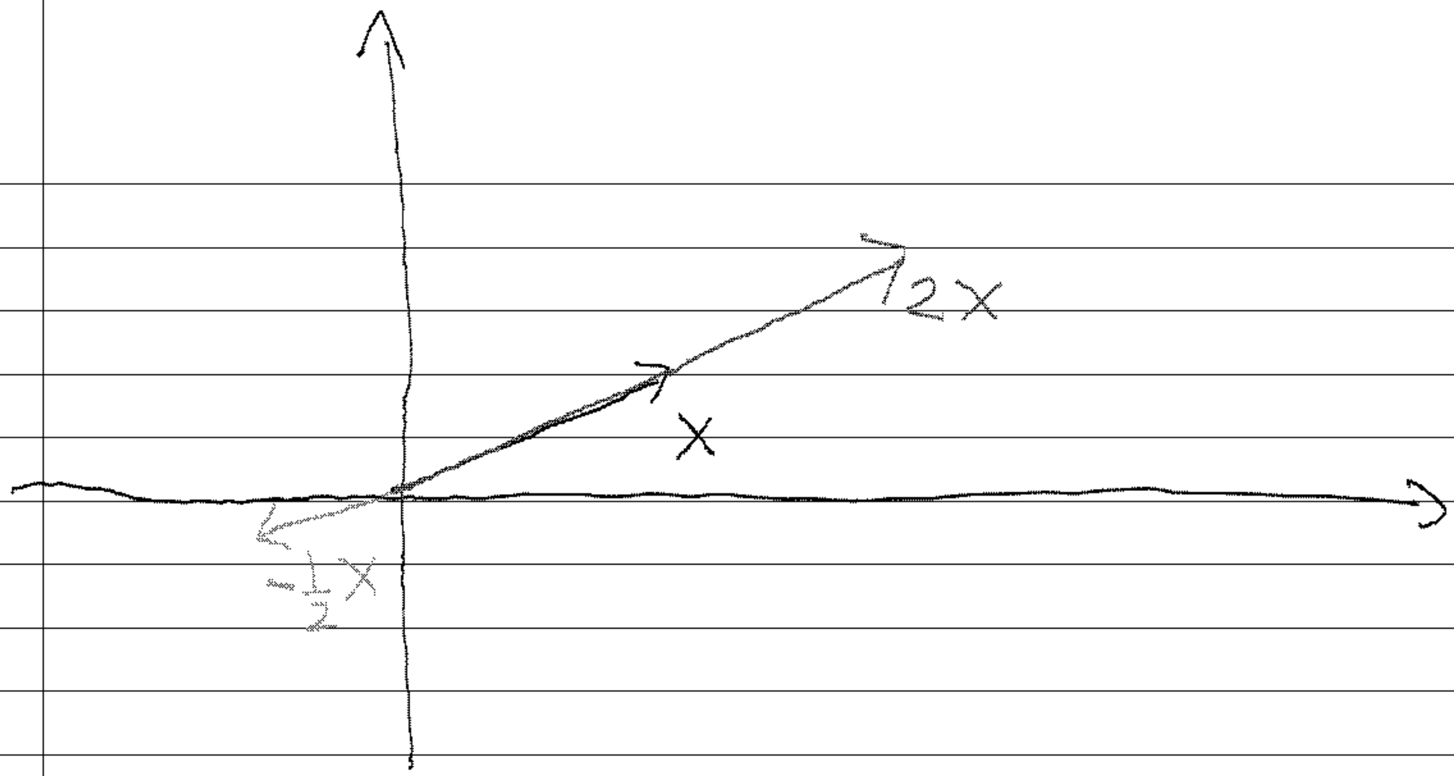
If $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$ or $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$ and $\lambda \in \mathbb{R}$

then $y = \lambda x \in \mathbb{R}^n$ where $y_i = \lambda x_i$

$y = (y_1, y_2, \ldots, y_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$

<u>Example</u> $3(-7, 5) = (-21, 15)$

Geometric interpretation of vector-scalar multiplication



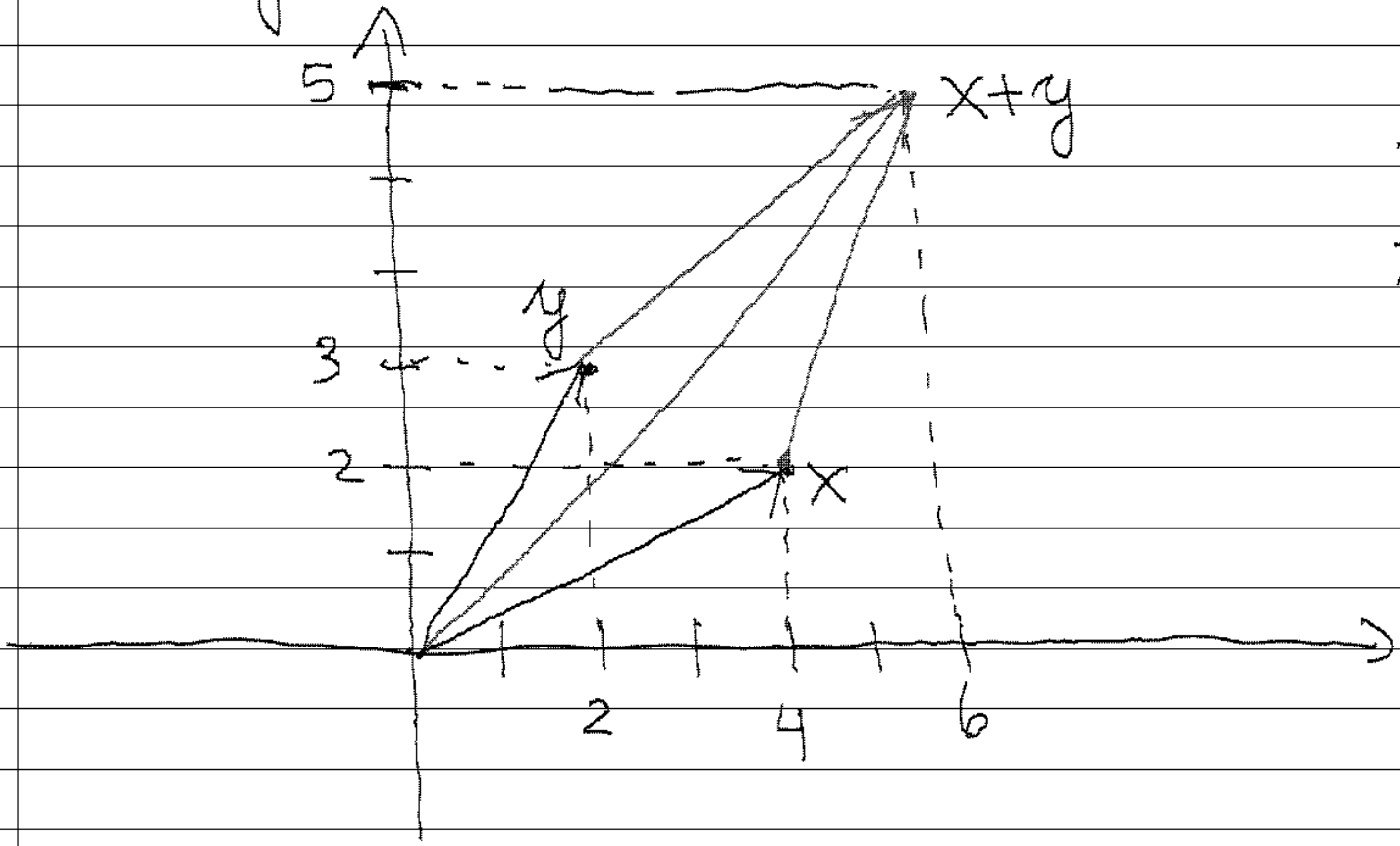
2) Addition of two vectors

If $x, y \in \mathbb{R}^n$, then $z = x+y \in \mathbb{R}^n$ and $z_i = x_i + y_i$

<u>Example</u> $(3,2,-1)+(5,1,7)=(8,3,6)$

Geometric interpretation of the addition of vectors

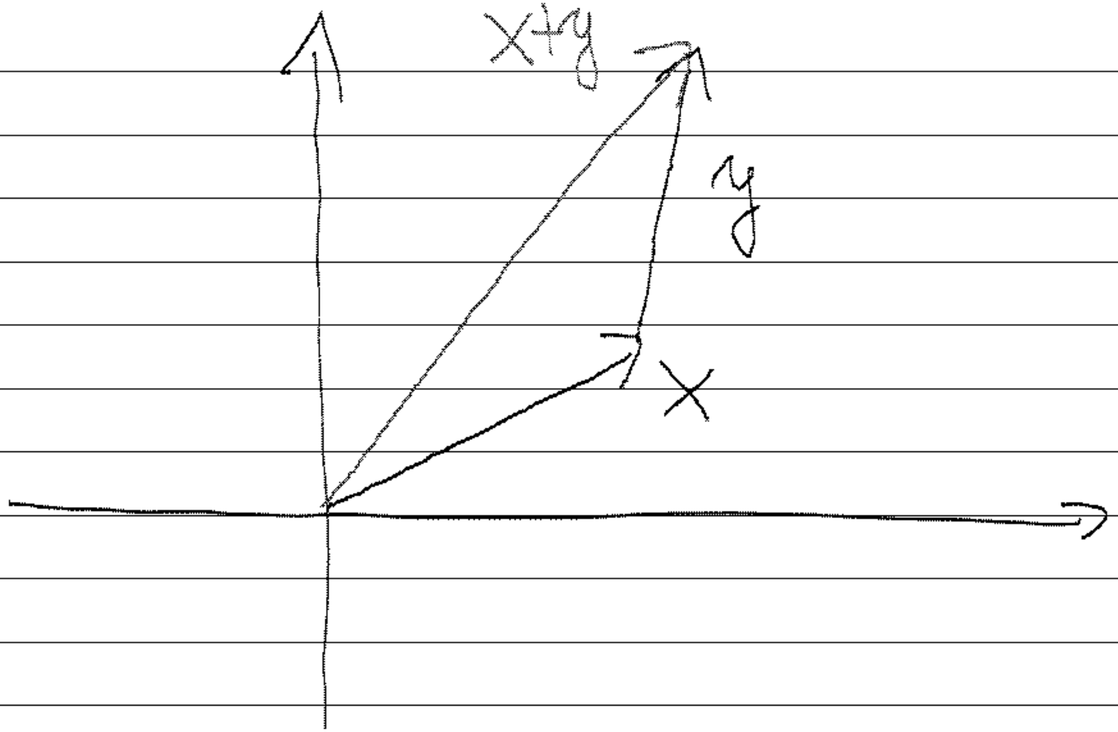
Parellogram rule



$x = (4,2)$

$y = (2,3)$

$x + y = (6,5)$



Properties: $x, y, z \in \mathbb{R}^n \quad \lambda, \beta \in \mathbb{R}$

1) $x+y = y+x$

2) $x+(y+z) = (x+y)+z$

3) $x+0=x$ $\quad 0=(0,0,\dots,0)\in\mathbb{R}^n$

4) $\lambda(x+y)=\lambda x+\lambda y$

5) $(\lambda+\beta)x=\lambda x+\beta x$

6) $\lambda(\beta x)=(\lambda\beta)x$

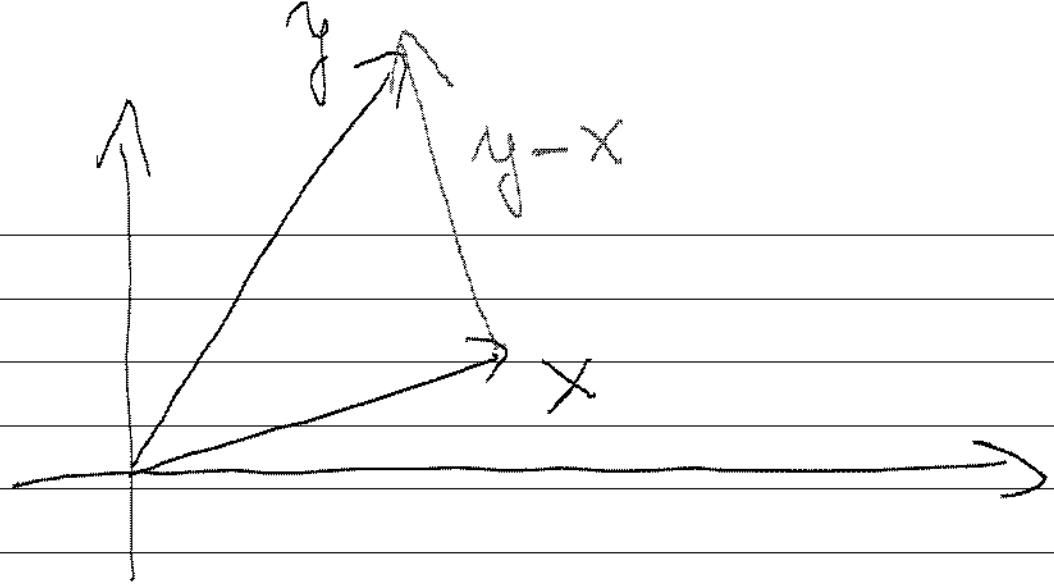
7) $1x=x$

8) $0x=0$

Notation $\quad -x=(-1)x$

$x-y=x+(-y)=x+(-1)y$

Geometric interpretation of substraction



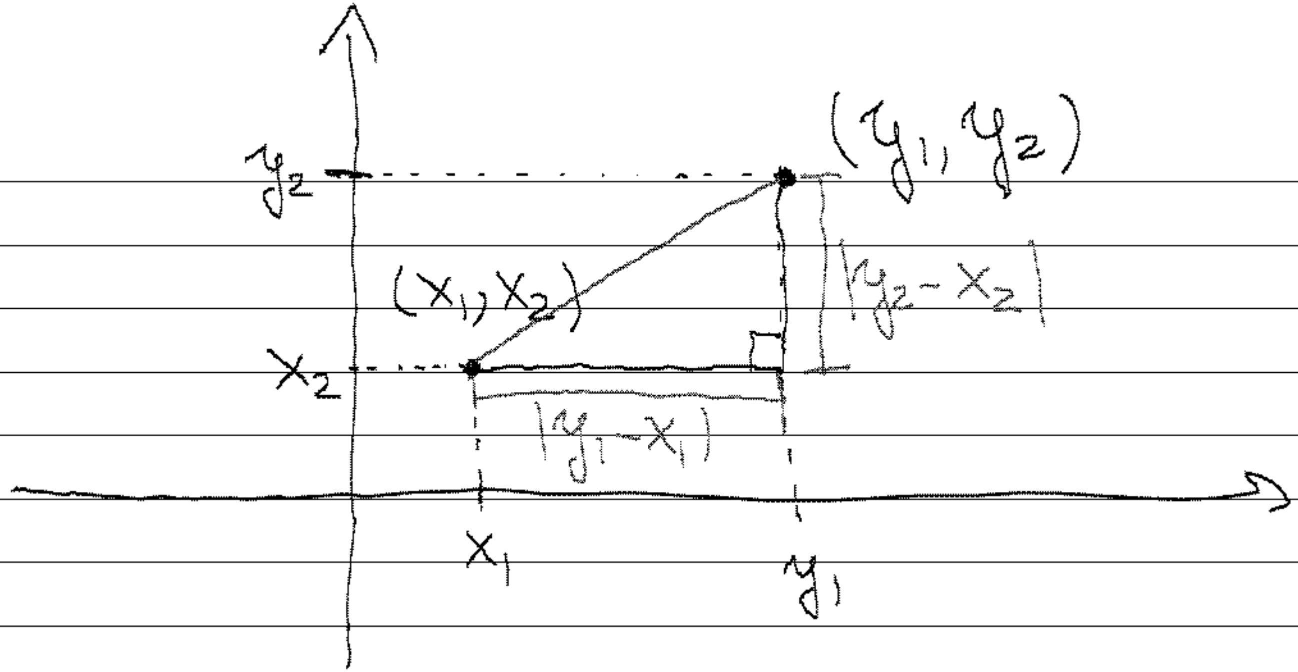
Note: $x+(y-x)=y$

<u>Notation</u> 1) i, j, k in $\mathbb{R}^3$ $i=(1,0,0)$ $j=(0,1,0)$ $k=(0,0,1)$

$i, j \in \mathbb{R}^2$ $i=(1,0)$ $j=(0,1)$

2) $\langle x_1, x_2, \ldots, x_n \rangle = (x_1, x_2, \ldots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

<u>Distance between vectors</u>



distance between x and y $= \sqrt{(y_1-x_1)^2+(y_2-x_2)^2}$

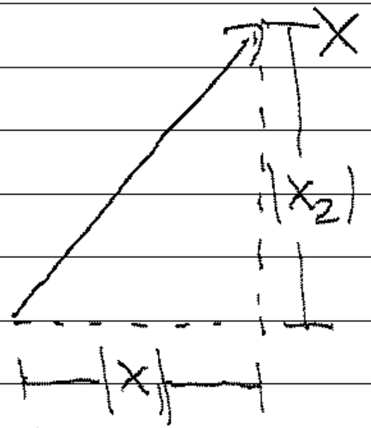
Def: If $x, y \in \mathbb{R}^n$, the distance between x and y is

$$\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+\cdots+(y_n-x_n)^2}$$

Example distance between $(1,-2,0)$ and $(3,5,1)$

$$\sqrt{(3-1)^2+(5-(-2))^2+(0-1)^2} = \sqrt{4+49+1} = \sqrt{54}$$

<u>Length of a vector</u> $\quad x \in \mathbb{R}^2$



length of $x = \sqrt{x_1^2+x_2^2}$

If $x \in \mathbb{R}^n$, the length of $x = \sqrt{x_1^2+x_2^2+\dots+x_n^2}$

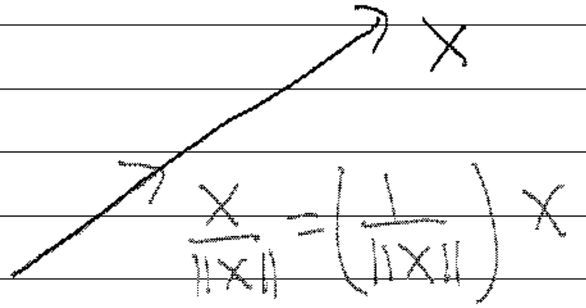
<u>Notation</u> $\|x\|$ = length of x = norm of x

<u>Obs</u>: $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

$$\|\lambda x\| = \sqrt{(\lambda x_1)^2+(\lambda x_2)^2+\dots+(\lambda x_n)^2} = |\lambda|\sqrt{x_1^2+x_2^2+\dots+x_n^2} = |\lambda|\,\|x\|$$

Notation: x is said to be a unit vector if $\|x\|=1$

Obs: $\left\| \frac{x}{\|x\|} \right\| = \left\| \frac{1}{\|x\|} x \right\| = \frac{1}{\|x\|} \|x\| = 1$



Given any vector x, with $x \neq 0$, the vector $\frac{x}{\|x\|}$ has norm 1 and points in the same direction as x.

Notation: Let $x_1, x_2, \ldots, x_r \in \mathbb{R}^n$. We say that y is a linear combination of $x_1, x_2, \ldots, x_r$ if there exists $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{R}$ such that $y = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_r x_r$

Dot Product

Def: $x, y \in \mathbb{R}^n$ the dot product of x and y is

$$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

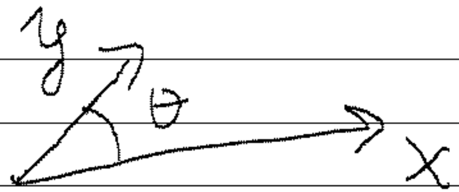
Example $(1,-1) \cdot (2,3) = 1(2) + (-1)3 = -1$

Properties of dot products $x, y, z \in \mathbb{R}^n$ $\lambda \in \mathbb{R}$

1) $x \cdot y = y \cdot x$
2) $x \cdot (y+z) = (x \cdot y) + (x \cdot z)$
3) $(\lambda x) \cdot y = \lambda (x \cdot y)$
4) $x \cdot x = x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$ and $x \cdot x = 0$ if and only if $\lambda = 0$

5) $x \cdot x = \|x\|^2$

<u>Obs</u>: Let $x, y \in \mathbb{R}^n$ $n = 2$ or $n = 3$



θ the angle between x and y

$0 \leq \theta \leq \pi$

Then, from the law of cosines, it follows that

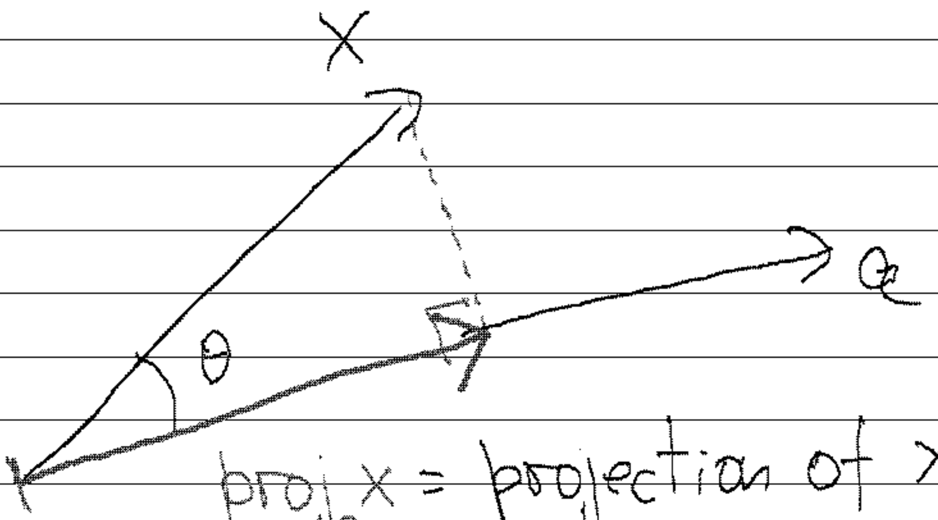
$$\boxed{x \cdot y = \|x\| \, \|y\| \cos\theta}$$

$$\cos\theta = \frac{x \cdot y}{\|x\| \, \|y\|} \quad \text{if } x \neq 0 \text{ and } y \neq 0$$

We use this as definition if $x, y \in \mathbb{R}^n$ $n \geq 4$

Def: x and y are orthogonal if $x \cdot y = 0$

Def



$\text{proj}_a x$ = projection of x onto a

$$\text{proj}_a x = \lambda a$$

$$\|\text{proj}_a x\| = |\lambda| \, \|a\| = |\cos\theta| \, \|x\| = \frac{|a \cdot x|}{\|a\| \, \|x\|} \|x\| = \frac{|a \cdot x|}{\|a\|}$$

$$\lambda = \frac{a \cdot x}{\|a\|^2}$$

$$\boxed{\operatorname{proj}_a x = \left(\frac{a \cdot x}{a \cdot a}\right) a}$$

We did sections 7.1, 7.2 and 7.3