Note Title 5/13/2015

Cross products

Determinants $a, b, c, d \in \mathbb{R}$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Example $\det \begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix} = 3(-2) - 1(4) = -10$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & k \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & k \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} =$$

Example $\det \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} = 1 \det \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - (-2) \det \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} +$

$0 \det \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} = 1\big(1(1) - 0(2)\big) + 2\big(3(1) - (-2)2\big) = 1 + 2(7) = 15$

<u>Cross product</u> $i = (1,0,0)$ $j = (0,1,0)$ $k = (0,0,1)$

$a = (a_1, a_2, a_3)$ $b = (b_1, b_2, b_3)$

The cross product of a and b is

$a \times b = \det \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = i \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - j \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} +$

$$+ k \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

Example: $a = (4, -2, 5)$ $b = (3, 1, -1)$

$$a \times b = \det \begin{bmatrix} i & j & k \\ 4 & -2 & 5 \\ 3 & 1 & -1 \end{bmatrix} = i \det \begin{bmatrix} -2 & 5 \\ 1 & -1 \end{bmatrix} - j \det \begin{bmatrix} 4 & 5 \\ 3 & -1 \end{bmatrix} + k \det \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} =$$

$$= i(-3) - j(-19) + k\,10 = -3i + 19j + 10k = -3(1,0,0) + 19(0,1,0) + 10(0,0,1) = (-3, 19, 10)$$

Properties $a, b, c \in \mathbb{R}^3$ $\lambda \in \mathbb{R}$

1) $a \times b = -b \times a$

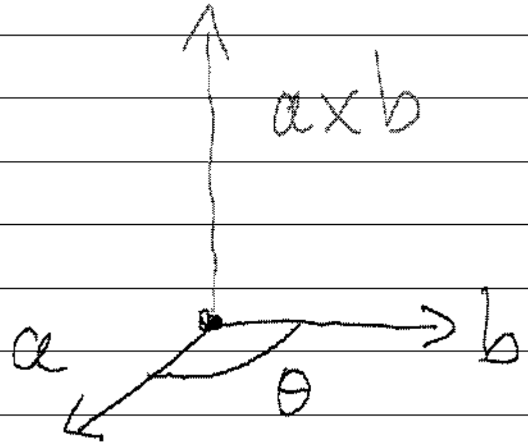
2) $a \times (b+c) = a \times b + a \times c$

3) $a \times (\lambda b) = \lambda (a \times b)$

4) $a \times a = 0$

5) $a \cdot (a \times b) = b \cdot (a \times b) = 0$

<u>Obs</u>; $a \times b$ is orthogonal to any vector in the plane that contains a, b and the origin 0

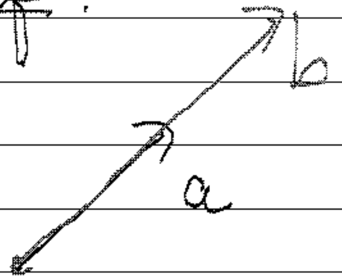


<u>Right</u> <u>hand</u> <u>rule</u> (book)

Theorem If θ is the angle between a and b, then

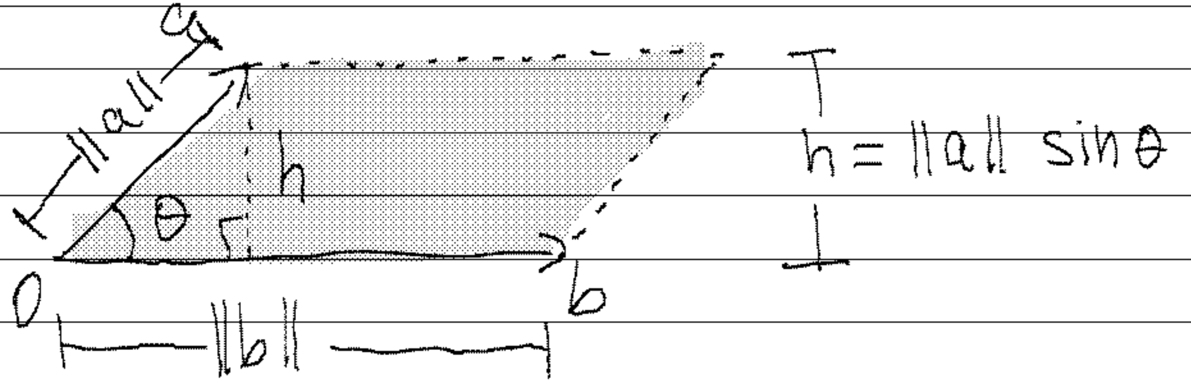
$$\|a\times b\| = \|a\|\|b\| \sin\theta$$

Def: Let $a \neq 0$ and $b \neq 0$, $a, b \in \mathbb{R}^n$. We say that they are parallel if there exists $\lambda \in \mathbb{R}$ such that $b = \lambda a$

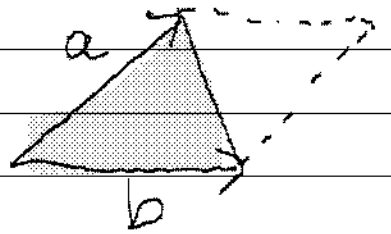


Obs: a and b are parallel if $a \times b = 0$ $\quad (a, b \in \mathbb{R}^3)$

Areas

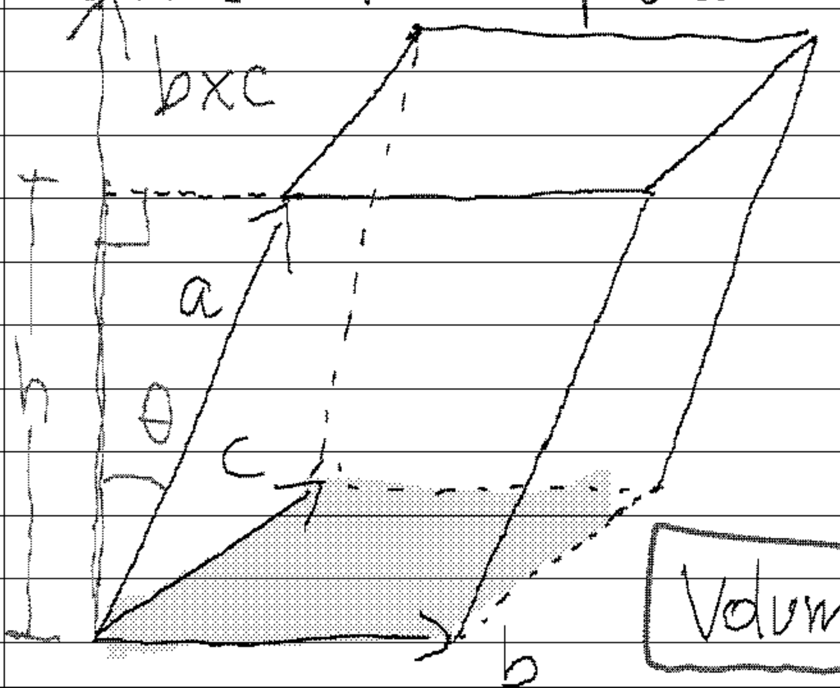


Area = base x height = $\|b\| \, \|a\| \sin\theta = \|a \times b\|$



Area of triangle = $\frac{1}{2} \|a \times b\|$

Volume of a parallelepiped



Volume = base x height

base = $\|b \times c\|$

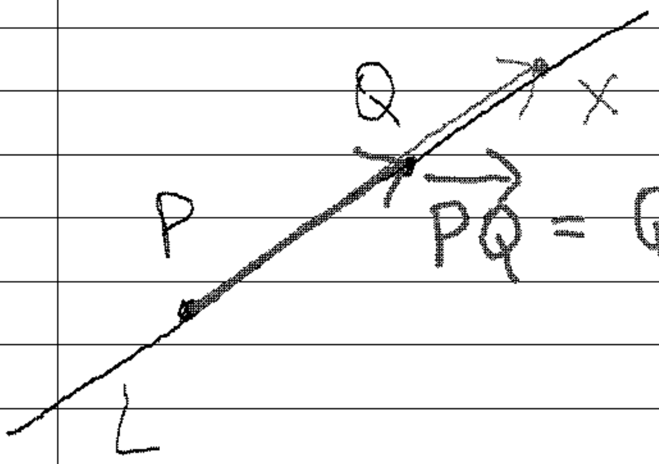
height = $h = \|a\| \, |\cos\theta| = \|a\| \dfrac{|a \cdot (b \times c)|}{\|a\| \, \|b \times c\|}$

$$\boxed{\text{Volume} = \|b \times c\| \, \|a\| \, \underline{|a \cdot (b \times c)|} = |a \cdot (b \times c)|}$$

$\|a\|\|b \times c\|$

<u>Obs</u> $a \cdot (b \times c) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

<u>Lines and planes in $\mathbb{R}^3$</u> L the line that contains P and Q



$\overrightarrow{PQ} = Q - P$

X belongs to the line L if and only if $\boxed{X = P + \lambda(Q-P)}$ for some $\lambda \in \mathbb{R}$

this is called a vector equation of the line

<u>Example</u> Find a vector equation for the line through $(2,-1,8)$ and $(5,6,-3)$

$$X = \boxed{(x_1, x_2, x_3)} = \underline{(2,-1,8) + \lambda\left((5,6,-3) - (2,-1,8)\right)} =$$

$$= \boxed{(2,-1,8) + \lambda(3,7,-11)}$$

<u>Parametric equations of a line</u>: As before, but just write component by component. In the example

$x_1 = 2 + 3\lambda$

$x_2 = -1 + 7\lambda$

$x_3 = 8 - 11\lambda$

<u>Symmetric equations</u> $\quad x_0, y_0, z_0, a_1, a_2, a_3 \in \mathbb{R}$

$(x, y, z) = (x_0, y_0, z_0) + t\,(a_1, a_2, a_3)$

$$t = \boxed{\frac{x-x_0}{a_1} = \frac{y-y_0}{a_2} = \frac{z-z_0}{a_3}}$$ these are called symmetric equations

Example: Find symmetric equations for the line through $(5,3,1)$ and $(2,1,1)$

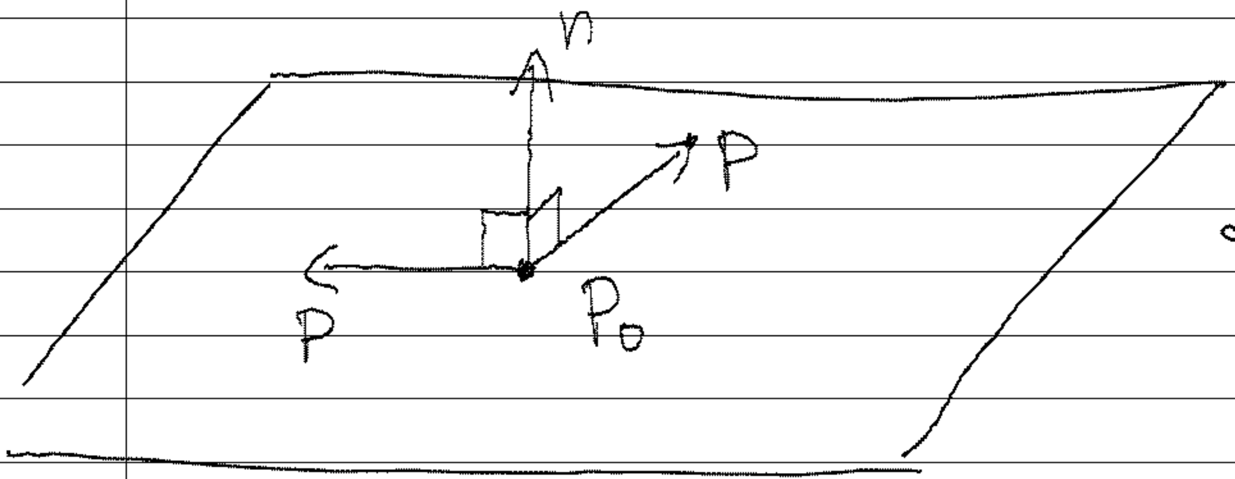
$(x_0, y_0, z_0) = (5,3,1)$ $\quad$ $(a_1, a_2, a_3) = (2,1,1)-(5,3,1) = (-3,-2,0)$

$$\boxed{\frac{x-5}{(-3)} = \frac{y-3}{(-2)} \qquad z = 1}$$

$(x,y,z) = (x_0, y_0, z_0) + \lambda\,(a_1, a_2, a_3)$

$(x,y,z) = (5,3,1) + \lambda\,(-3,-2,0)$

Planes in $\mathbb{R}^3$



P_0 a point in the plane

n a vector such that $\hat{n}$ is orthogonal to $\overrightarrow{P_0P}$ for any P in the plane.

P is in the plane if and only if $n \cdot \overrightarrow{P_0P} = 0$

If $P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$ $n = (a, b, c)$

$$n \cdot \overrightarrow{P_0P} = \boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$ Cartesian equation

Example: Find an equation of the plane with normal vector (perpendicular) $n = (2, 8, -5)$ containing the point $(4, -1, 3)$

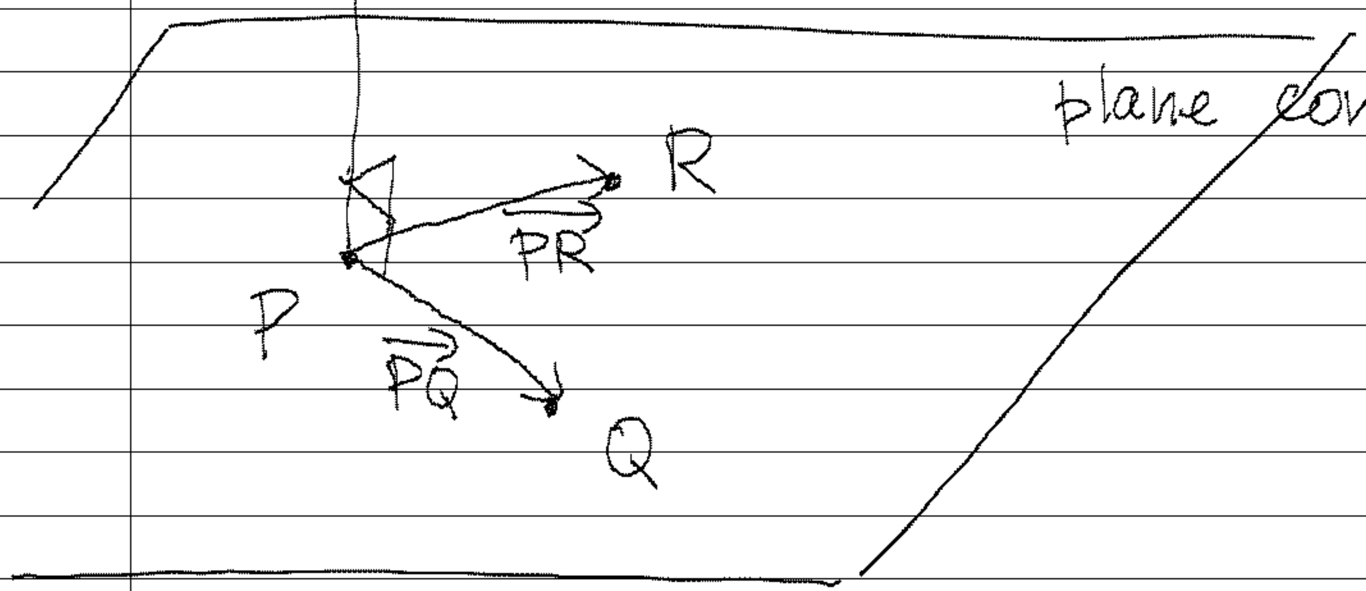
$$\boxed{2(x-4) + 8(y+1) - 5(z-3) = 0}$$

Obs: $\boxed{ax + by + cz + d = 0}$ a, b and c not all zero

a, b, c and d given. This is an equation of a plane with normal vector $n = (a, b, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Obs: Three points that do not belong to the same line determine a plane

$\overrightarrow{PQ}\times\overrightarrow{PR}=n$ this is a vector perpendicular to the plane that contains P, Q & R



<u>Ex</u>: Find an equation of the plane containing $\underset{P}{\underset{\shortparallel}{(1,0,-1)}}$, $\underset{Q}{\underset{\shortparallel}{(3,1,4)}}$ and $(2,-2,0)=R$

$$(3,1,4)-(1,0,-1)=(2,1,5)=\overrightarrow{PQ}$$

$$(2,-2,0)-(1,0,-1)=(1,-2,1)=\overrightarrow{PR}$$

$$\det\begin{bmatrix} i & j & k \\ 2 & 1 & 5 \\ 1 & -2 & 1 \end{bmatrix} = i(11)-j(-3)+k(-5)=(11,3,-5)$$

$$\boxed{11(x-1)+3y-5(z+1)=0}$$

Vector spaces

Definition: A vector space is a set V with two operations: a vector addition and a scalar multiplication, where the following is satisfied:

1) $x, y \in V$, then $x+y \in V$

2) $x+y = y+x$

3) $x+(y+z) = (x+y)+z$

4) there exists $0 \in V$ such that $x+0 = x$ for all $x \in V$

5) $x \in V$ $\lambda \in \mathbb{R}$ then $\lambda x \in V$

6) $\lambda(x+y) = \lambda x + \lambda y$

7) $(\lambda+\beta)x = \lambda x + \beta x$

8) $(\lambda\beta)x = \lambda(\beta x)$

9) $1x = x$

Examples: 1) $\mathbb{R}^n$

2) Polynomials

3) $C[a,b]$ set of continuous real valued functions defined on the interval $[a,b]$