SHADOWS AND CONVEXITY OF SURFACES

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Abstract. We study the geometry and topology of immersed surfaces in Euclidean 3-space whose Gauss map satisfies a certain two-piece-property, and solve the “shadow problem” formulated by H. Wente.

1. Introduction

Let $M$ be a closed oriented 2-dimensional manifold, $f : M \to \mathbb{R}^3$ be a smooth immersion into Euclidean 3-space, and $n : M \to S^2$ be a unit normal vectorfield, or the Gauss map, induced by $f$. Then for every unit vector $u \in S^2$ (corresponding to the direction of light) the shadow, $S_u$, is defined by

$$S_u := \{ p \in M : \langle n(p), u \rangle > 0 \},$$

where $\langle \cdot, \cdot \rangle$ is the standard innerproduct. If $f$ is a convex embedding, i.e., $f$ maps $M$ homeomorphically to the boundary of a convex body, then it is intuitively clear that $S_u$ is a connected subset of $M$ for each $u$. In 1978, motivated by problems concerning the stability of constant mean curvature surfaces [17], H. Wente appears to have been the first person to study the converse of this phenomenon, which has since become known as the “shadow problem” [13]: Does connectedness of the shadows $S_u$ imply that $f$ is a convex embedding? In this paper we prove:

**Theorem 1.1.** $f$ is a convex embedding if and only if, for every $u \in S^2$, $S_u$ is simply connected.

Furthermore we show that the additional condition implied by the word “simply” in the above theorem is necessary:

**Theorem 1.2.** There exists a smooth embedding of the torus, $f : S^1 \times S^1 \to \mathbb{R}^3$, such that for all $u \in S^2$, $S_u$ is connected.

Thus, connectedness of the shadows in general is not strong enough to ensure convexity or even determine the topology; however, we can show:

**Theorem 1.3.** If $M$ is topologically a sphere, and, for every $u \in S^2$, $S_u$ is connected, then $f$ must be a convex embedding.

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In short, the answer to the above question is yes, provided that either the shadows are simply connected, or \( M \) is a sphere; otherwise, the answer is no. This settles Wente’s shadow problem in 3-space. See [7] and [5] for motivations behind this problem and relations to constant mean curvature surfaces.

**Note 1.4.** The immersion \( f : M \to \mathbb{S}^2 \) has connected shadows if and only if for every great circle \( C \subset \mathbb{S}^2 \), \( n^{-1}(\mathbb{S}^2 - C) \) has exactly two components. That is, the Gauss map satisfies a *two-piece-property* [3] similar to that formulated by T. Banchoff [2], and further developed by N. Kuiper [12].

**Note 1.5.** For a great circle \( C \subset \mathbb{S}^2 \), the number of components of \( n^{-1}(\mathbb{S}^2 - C) \) has been called the *vision number* with respect to a direction perpendicular to \( C \). This terminology is due to J. Choe, who conjectured [5, p. 210] that there always exists a direction with respect to which the vision number of \( f : M \to \mathbb{R}^3 \) is greater than or equal to \( 4 - \chi(M) \) where \( \chi \) is the Euler characteristic. Theorem 1.2 gives a counterexample to this conjecture.

2. Regularity of Horizons and Shadow Boundaries

First we need to establish some basic regularity results regarding the generic behavior of shadows. For each \( u \in \mathbb{S}^2 \), define the *shadow function* \( \sigma_u : M \to \mathbb{R} \) by

\[
\sigma_u(p) := \langle n(p), u \rangle.
\]

\( H_u := \sigma_0^{-1}(0) \) is called the *horizon* [5] in the direction \( u \). It is easy to see that in general \( \partial S_u \neq H_u \neq \partial S_{-u} \), where \( \partial \) denotes the boundary; however, using Sard’s theorem, we can show

**Proposition 2.1.** For almost all \( u \in \mathbb{S}^2 \) (in the sense of Lebesgue measure) \( H_u \) is a regular curve. Thus for these \( u \), both \( \partial S_u \) and \( \partial S_{-u} \) are regular curves as well. Further, if \( H_u \) is connected, then \( \partial S_u = H_u = \partial S_{-u} \).

We say that \( \Gamma \subset M \) is a *regular curve* if for each \( p \in \Gamma \) there is an open neighborhood \( U \) of \( p \) in \( M \) and a homeomorphism \( \varphi : U \to \mathbb{R}^2 \) such that \( \varphi(U \cap \Gamma) = \mathbb{R} \). In particular, unless stated otherwise, a regular curve needs not be differentiable.

**Proof.** Let \( T_pM \) be the tangent plane of \( M \) at \( p \) which we identify with a subspace of \( \mathbb{R}^3 \) (by identifying \( T_pM \) with \( f_\ast(T_pM) \), and parallel translating the elements of \( f_\ast(T_pM) \) to the origin in \( \mathbb{R}^3 \); \( f_\ast \) denotes the differential of \( f \)). Let \( UTM := \{ (p, u) : p \in M, u \in T_pM, \|u\| = 1 \} \) denote the unit tangent bundle of \( M \), and \( \tau \) be the mapping given by

\[
UTM \ni (p, u) \mapsto \tau(u) \in \mathbb{S}^2.
\]

By Sard’s Theorem almost every \( u \in \mathbb{S}^2 \) is a regular value of \( \tau \); consequently, for such \( u \), \( \tau^{-1}(u) \) is a regular curve in \( UTM \).

Now let \( \pi \) be the mapping defined by

\[
UTM \ni (p, u) \mapsto \pi(u) \in M,
\]

\( \pi(u) := \varphi^{-1}(\tau(u)) \), where \( \varphi \) is a regular curve.
and let \( u \) be a regular value of \( \tau \). Note that \( \pi \) is injective on \( \tau^{-1}(u) \). As \( \tau^{-1}(u) \) is compact, this implies that \( \tau^{-1}(u) \to M \) is an embedding. Further note that
\[
\pi(\tau^{-1}(u)) = \{ p \in M : u \in T_p M \} = \{ p \in M : \langle n(p), u \rangle = 0 \} = H_u.
\]
Thus \( H_u \) is a regular curve. But then, it follows that \( \partial S_u \) and \( \partial S_{-u} \) are each open in \( H_u \), which yields that \( \partial S_u \) and \( \partial S_{-u} \) are both regular curves as well. Finally, since these shadow boundaries are also closed in \( H_u \), it follows that whenever \( H_u \) is connected we have \( \partial S_u = H_u = \partial S_{-u} \).

**Note 2.2.** Suppose there is an open set \( U \subset \mathbb{S}^2 \), such that, for all \( u \in U \), both \( S_u \) and \( S_{-u} \) are simply connected. Then \( M \) is homeomorphic to \( \mathbb{S}^2 \); because, by the above proposition, there exists a \( u_0 \in U \) such that \( H_{u_0} \) is a regular curve. Consequently the closures \( \overline{S}_{u_0} \) and \( \overline{S}_{-u_0} \) are homeomorphic to disks. Further, since by assumption \( M - H_{u_0} \) is made up of a pair of simply connected components, \( H_{u_0} \) is connected. Thus by the above proposition \( \partial S_{-u_0} = \partial S_{u_0} \). So \( M \) is homeomorphic to a pair of disks glued together along their boundaries.

By *smooth* we mean differentiable of class \( C^\infty \), and for convenience we always assume that the immersion \( f : M \to \mathbb{R}^3 \) is smooth, though in this paper it is enough that \( f \) be \( C^3 \).

**Note 2.3.** The embedding \( \pi : \tau^{-1}(u) \to M \) in the above proposition is smooth, when \( u \) is a regular value of \( \tau \). In particular, \( H_u \) is smooth for almost all \( u \in \mathbb{S}^2 \). To see this let \( (p, u) \in \tau^{-1}(u) \). Then \( u \in T_p M \). Let \( v \in T_p M \) with \( \langle u, v \rangle = 0 \). Then \( c(t) := (p, \cos(t)u + \sin(t)v) \) parameterizes the fiber \( UT_p M \) of the unit tangent bundle. Note that
\[
\tau_{\tau^{-1}(u)}(c'(0)) = \frac{d}{dt} \tau\left(p, \cos(t)u + \sin(t)v\right)\big|_{t=0} = v \neq 0.
\]
On the other hand,
\[
T\left(\tau^{-1}(u)\right) = \{ X \in T_{\tau^{-1}(u)}(UT M) : \tau_{\tau^{-1}(u)}(X) = 0 \}.
\]
Thus \( c'(0) \notin T_{\tau^{-1}(u)}(\tau^{-1}(u)) \), which implies that \( \tau^{-1}(u) \) is never tangent to any of the fibers \( UT_p M \) of the unit tangent bundle. So \( \tau_{\tau^{-1}(u)} \) is a smooth immersion.

Next we need a local regularity result for the horizons and shadow boundaries. The *Gaussian curvature* \( K : M \to \mathbb{R} \) is defined by \( K(p) := \det(n_*(p)) \).

**Proposition 2.4.** If \( K(p) \neq 0 \) for some \( p \in M \), then there exists a neighborhood \( U \) of \( p \) such that for all \( u \in T_u M \), \( H_u \cap U \) is a smooth regular curve and \( \partial S_u \cap U = H_u \cap U = \partial S_{-u} \cap U \).

*Proof.* Since \( \det(n_*) = K(p) \neq 0 \), then, by the inverse function theorem, \( n \) is a diffeomorphism between small neighborhoods \( U \) of \( p \) in \( M \) and \( V \) of \( n(p) \) in \( \mathbb{S}^2 \). Let \( \mathbb{S}^2_u := \{ x \in \mathbb{S}^2 : \langle x, u \rangle > 0 \} \). Then \( \partial \mathbb{S}^2_u = \partial \mathbb{S}^2_{-u} \) is a regular curve. Thus, since \( S_u = n^{-1}(\mathbb{S}^2_u) \) and \( S_{-u} = n^{-1}(\mathbb{S}^2_{-u}) \), the proof follows. \( \square \)
Note 2.5. If $K(p) = 0$, then $H_u$ may not be regular for all $u \in T_p M$; however, typically $H_u$ will be regular for most $u \in T_p M$; because, for $u \in T_p M$, the differential of $\sigma_u$ at $p$ is given by

$$\left(\delta \sigma_u\right)_p(\cdot) = \left\langle \cdot, n_u(p) \right\rangle.$$ 

So if $n_u(p) \neq 0$, e.g., $u$ is not an asymptotic direction, then $\delta \sigma_u$ is nonzero at $p$. Consequently, by the implicit function theorem, $\sigma_u^{-1}(\sigma_u(p)) = \sigma_u^{-1}(0) = H_u$ is a smooth regular curve near $p$.

3. Critical Points of Height Functions

The next set of preliminary results we need involves some basic applications of Morse theory [14]. For every $u \in S^2$, let the height function $h_u : M \to \mathbb{R}$, associated to the immersion $f : M \to \mathbb{R}^3$, be defined by

$$h_u(p) := \left\langle f(p), u \right\rangle.$$ 

Recall that $p$ is a critical point of $h_u$ if the differential map $(dh_u)_p : T_p M \to \mathbb{R}$ is zero. Since $(dh_u)_p(\cdot) = \left\langle \cdot, u \right\rangle$, it follows that $p$ is a critical point of $h_u$ if and only if $u = \pm n(p)$. If all of its critical points are nondegenerate, $h_u$ is a Morse function.

Lemma 3.1. (i) $h_u$ is a Morse function if and only if $K \neq 0$ at all critical points of $h_u$. (ii) $h_u$ is a Morse function for almost all $u \in S^2$. (iii) The set $U \subset S^2$ such that for all $u \in U$ $h_u$ is a Morse function is open.

Though the above is fairly well-known (e.g. see [3, pp. 11–12]), we include a brief proof for completeness.

Proof. If $p$ is a critical point of $h_u$, then, as a standard computation shows, the Hessian of $h_u$ is given by

$$\text{Hess } h_u(\cdot, \cdot) = \pm \left\langle \cdot, n_u(\cdot) \right\rangle.$$ 

Thus $h_u$ is a Morse function if and only if at each critical point $p$, $K(p) = \det(n_u) \neq 0$. This is equivalent to requiring that both $u$ and $-u$ be regular values of $n$, because $p$ is a critical point of $h_u$ if and only if $u = \pm n(p)$. Let $U \subset S^2$ be the set of all such values. Then, by Sard’s theorem, $S^2 - U$ has measure zero. Further, since $M$ is compact, and the set of critical points of $n$ is closed, it follows that the set of critical values of $n$ is closed as well, so $U$ is open.

The following is implicit in a paper of Chern and Lashof [4].

Lemma 3.2. If $f$ is not a convex embedding, then there exists a Morse height function $h_u$ with at least 3 critical points.

Proof. Let $\#C(h_u)$ denote the number of critical points of $h_u$. Since $p$ is a critical point of $h_u$ if and only if $n(p) = \pm u$, we have:

$$\int_{S^2} \#C(h_u) \, du = \int_{S^2} \#n^{-1}(u) \, du = 2 \int_M |\det(n)| \, dV = 2 \int_M |K| \, dV.$$
The second equality above is just an application of the area formula [6, Thm 3.2.3], where $dV$ denotes the volume element on $M$. Suppose that $f$ is not a convex embedding. Then, by a well-known theorem of Chern and Lashof [4],

$$\int_M |K| dV > 4\pi.$$ 

Combining the above expressions yields a lower bound for the average number of critical points:

$$\frac{1}{4\pi} \int_{S^2} #C(h_u) du > 2.$$ 

So since, by Lemma 3.1, $h_u$ is a Morse function for almost every $u \in S^2$, it follows that there exists a Morse function such that $#C(h_u) > 2$. \hfill \square

4. Triplets on the Boundaries of Simply Connected Domains

Here we develop some elementary topological methods whose motivation will become more clear in the next section.

**Definition 4.1.** By a domain we mean a connected open subset $\Omega \subset M$. We say $\Omega$ is adjacent to a triplet of points $\{p_1, p_2, p_3\} \subset M$ if $p_i \in \partial \Omega$. $\Omega$ is regular near $p_i$ if there are open neighborhoods $U_i$ of $p_i$ and homeomorphisms $\varphi_i : U_i \to \mathbb{R}^2$ which map $U_i \cap \Omega$ into the upper half-plane. A simple closed curve $T \subset \overline{\Omega}$ is a triangle of $\Omega$ (with vertices at $\{p_1, p_2, p_3\}$) if $p_i \in T$, and $T - \{p_1, p_2, p_3\} \subset \Omega$.

The following lemma, though quite elementary, is more subtle than it might at first appear (see Note 4.3).

**Lemma 4.2.** Every domain $\Omega$ adjacent to $\{p_1, p_2, p_3\}$ admits a triangle. Further if $\Omega$ is simply connected and regular near $p_i$, then any pair of such triangles may be homotoped to each other through a family of triangles of $\Omega$.

**Proof.** Since $\Omega$ is open and connected, there exists a regular arc $A_{12} \subset \Omega$ whose end points are $p_1$ and $p_2$. Since $A_{12}$ is regular, there exists a component $(\Omega - A_{12})^+$ of $\Omega - A_{12}$ which contains $p_3$ in its closure. Let $A_{23} \subset (\Omega - A_{12})^+$ be a regular arc with end points on $p_2$ and $p_3$. Then, similarly, there exists a component $(\Omega - A_{23})^+$ of $(\Omega - A_{12})^+ - A_{23}$ which contains $p_1$ in its closure. Finally, let $A_{31} \subset ((\Omega - A_{12})^+ - A_{23})^+$ be a regular arc with end points at $p_3$ and $p_1$. The union of these three arcs, and their endpoints, gives the desired triangle.

Now suppose that $\Omega$ is simply connected and regular near $p_i$. Let $T$ and $T'$ be a pair of triangles of $\Omega$, and let $A_{12}$ and $A'_{12}$ be arcs of $T$ and $T'$ respectively which connect $p_1$ and $p_2$. Since $\Omega$ is regular near $p_1$, we may homotope $A_{12}$ (while keeping its end points fixed) by a small perturbation near $p_1$ so that $A_{12}$ and $A'_{12}$ coincide along a segment near $p_1$. Similarly, we may assume that they coincide near $p_2$ as well. Then it remains to homotope proper subarcs of $A_{12}$ and $A'_{12}$ which coincide at a pair of end points in $\Omega$. Since $\Omega$ is simply connected, these subarcs may be homotoped to each other while keeping the end points fixed. Thus $A_{12}$ and $A'_{12}$ are homotopic through a family of arcs of $\Omega$ with end points at $p_1$ and $p_2$. Other arcs
of \( T \) may be similarly homotoped to their counterparts in \( T' \), which completes the proof. \( \square \)

**Note 4.3.** Without the regularity assumption near \( p_i \), the second claim in the above lemma is not true in general: Suppose for instance that \( \Omega \subset \mathbb{R}^2 \) is an open disk of radius 1 centered at the origin, and with segment \([0,1] \) removed. Set \( p_1 = (0,0) \), \( p_2 = (1/2,0) \), and \( p_3 = (1,0) \). Then a triangle of \( \Omega \) which lies above the \( x \)-axis may not be homotoped to one lying below the \( x \)-axis.

**Proposition 4.4.** For a fixed orientation of \( M \), every simply connected domain \( \Omega \) which is adjacent to and regular near a triple of (distinct) points \( \{p_1, p_2, p_3\} \subset M \) uniquely determines a permutation \( \alpha_\Omega \) of \( \{p_1, p_2, p_3\} \) such that (i) if \( \Omega \) and \( \Omega' \) have a triangle in common, then \( \alpha_\Omega = \alpha_{\Omega'} \); and (ii) if \( \partial \Omega = \partial \Omega' \) is a regular curve, and \( \Omega \) and \( \Omega' \) are distinct, then \( \alpha_\Omega \neq \alpha_{\Omega'} \).

**Proof.** By Lemma 4.2 there exists a triangle \( T \) of \( \Omega \). \( T \) bounds a simply connected subdomain \( U \) of \( \Omega \). Since \( M \) is oriented, \( U \) inherits a preferred sense of orientation, which in turn induces an orientation, or a sense of direction, on \( T \). This direction induces a permutation of \( \{p_1, p_2, p_3\} \) in the obvious way: If as we move along \( T \) and pass \( p_1 \) we reach \( p_2 \) before reaching \( p_3 \), then we set the induced permutation \( \alpha_\Omega \) to be the cycle \( (p_1, p_2, p_3) \); otherwise, the induced permutation is the cycle \( (p_1, p_3, p_2) \).

It is clear that these permutations depend continuously on \( T \). Thus, since by Lemma 4.2, all triangles of \( \Omega \) are homotopic, it follows that \( \alpha_\Omega \) does not depend on the choice of \( T \) and is therefore well defined; and furthermore, if \( \Omega \) and \( \Omega' \) have a triangle in common then \( \alpha_\Omega = \alpha_{\Omega'} \).

Now suppose that \( \partial \Omega = \partial \Omega' \) is a regular curve, and \( \Omega \) and \( \Omega' \) are distinct. Then \( \Omega \) and \( \Omega' \) induce opposite orientations on \( \partial \Omega \) which in turn gives rise to distinct permutations of \( \{p_1, p_2, p_3\} \) (since \( \Omega \) is simply connected, \( \partial \Omega \) is connected). But by small perturbations, \( \partial \Omega \) may be homotoped to a triangle of \( \Omega \), just as well as it may be homotoped to a triangle of \( \Omega' \). Thus the orientations which \( \Omega \) and \( \Omega' \) induce on \( \partial \Omega \) are consistent with the orientations which \( \Omega \) and \( \Omega' \) induce on their own triangles respectively. So \( \alpha_\Omega \neq \alpha_{\Omega'} \). \( \square \)

5. **Proof of Theorem 1.1**

First we show that if \( f \) is a convex embedding, then \( S_u \) is simply connected for all \( u \in S^2 \). To see this let \( \Pi \) be a plane perpendicular to \( u \) and let \( \pi: \mathbb{R}^3 \to \Pi \) be the orthogonal projection. Then \( D := \pi(f(M)) \) is a convex subset of \( \Pi \) with interior points. In particular, \( \text{int}(D) \) is homeomorphic to an open disk. Since \( f(M) \) is convex and by definition \( \langle n(p), u \rangle > 0 \) for all \( p \in S_u \), it is not hard to verify that \( f(S_u) \) is a graph over \( \text{int}(D) \). Thus \( \pi \circ f: S_u \to \text{int}(D) \) is a homeomorphism.

Now we prove the other direction: Assume that for every \( u \in S^2 \), \( S_u \) is simply connected; we have to show that \( f \) is a convex embedding. The proof is by contradiction:

**Lemma 5.1.** If \( f \) is not a convex embedding, then there exists a pair of orthogonal vectors \( u_0, v_0 \in S^2 \) such that (i) \( h_{u_0} \) is a Morse function with at least 3 critical points, and (ii) \( \partial S_{v_0} = H_{v_0} = \partial S_{-v_0} \) is a regular curve.
Proof. By Lemma 3.2, there exists a unit vector \( u \in S^2 \) such that the corresponding height function \( h_u \) is a Morse function and has at least three critical points. Further, it follows from Lemma 3.1, that this \( u \) may be chosen from an open set \( U \subset S^2 \).

Let \( u^\perp := \{ u \in S^2 : \langle u, v \rangle = 0 \} \). Then \( U^\perp := \cup_{u \in U} u^\perp \) is open. Consequently, by Proposition 2.1, there exists a \( v_0 \in u^\perp \subset U^\perp \) such that \( H_{v_0} \) is a regular curve. Further, since the complement of \( H_{v_0} \) consists of a pair of simply connected domains, \( H_{v_0} \) is connected. Thus, again by Proposition 2.1, \( \partial S_{v_0} = H_{v_0} = \partial S_{-v_0} \) is a regular curve. \( \square \)

Let \( \widehat{v}_0 \in S^2 \) be a vector orthogonal to both \( u_0 \) and \( v_0 \), and set

\[
\nu(\theta) := \cos(\theta) v_0 + \sin(\theta) \widehat{v}_0.
\]

Let \( p_i, i = 1, 2, 3 \), be a fixed triple of (distinct) critical points of \( h_{u_0} \).

**Lemma 5.2.** For all \( \theta \in \mathbb{R} \), \( S_{\nu(\theta)} \) is a domain adjacent to and regular near \( p_i \).

**Proof.** If \( p_i \) is a critical point of \( h_{u_0} \), then \( n(p_i) = \pm u_0 \). So \( \sigma_{\nu(\theta)}(p_i) = \langle \nu(\theta), \pm u_0 \rangle = 0 \), which yields that \( p_i \in H_{\nu(\theta)} \). Since \( h_{u_0} \) is a Morse function, then, by Lemma 3.1, \( K(p_i) \neq 0 \). So by Proposition 2.4, there exists a neighborhood \( U_i \) of \( p_i \) such that \( \partial S_{v_0} \cap U_i = H_{\nu(\theta)} \cap U_i = \partial S_{-v_0} \cap U_i \), which completes the proof. \( \square \)

It now follows from Proposition 4.4 that each \( S_{\nu(\theta)} \) induces a permutation of \( \{ p_1, p_2, p_3 \} \) which we denote by \( \alpha_{\theta} := \alpha_{S_{\nu(\theta)}} \). Further, by the same proposition and since \( \partial S_{v_0} = \partial S_{-v_0} \) is a regular curve, it follows that \( \alpha_0 \neq \alpha_{\pi} \). On the other hand, letting Sym denote the symmetric group, we claim that the mapping

\[
\mathbb{R} \ni \theta \mapsto \alpha_{\theta} \in \text{Sym}(\{ p_1, p_2, p_3 \})
\]

is locally constant, which, since \([0, \pi] \) is connected, would imply that \( \alpha_0 = \alpha_{\pi} \). This contradiction, which would complete the proof, follows from Proposition 4.4 and the following:

**Lemma 5.3.** For each \( \theta_0 \in \mathbb{R} \) there exists an \( \epsilon > 0 \) such that if \( |\theta - \theta_0| < \epsilon \) then \( S_{\nu(\theta)} \) and \( S_{\nu(\theta_0)} \) have a common triangle (with vertices at \( \{ p_1, p_2, p_3 \} \)).

**Proof.** Recall that, since \( h_{u_0} \) is a Morse function, then, by Lemma 3.1, \( K(p_i) \neq 0 \) which yields that \( n \) is a local diffeomorphism at \( p_i \). Therefore, by Proposition 2.4, in a neighborhood \( W \) of \( \{ p_1, p_2, p_3 \} \), \( \partial S_{\nu(\theta)} = H_{\nu(\theta)} = n^{-1}(v^+(\theta)) \) where \( v^+(\theta) \) denotes the great circle in \( S^2 \) orthogonal to \( v(\theta) \). So, since \( v^+(\theta) \) depends continuously on \( \theta \), it follows that, in \( W \), \( \partial S_{\nu(\theta)} \) depends continuously on \( \theta \) as well.

Let \( T \) be a triangle of \( S_{\nu(\theta_0)} \). Since \( S_{\nu(\theta_0)} \) is open, after a perturbation of \( T \) we may assume that the arcs of \( T \) are smooth and meet \( \partial S_{\nu(\theta_0)} \) transversely (recall that, by Proposition 2.4, \( \partial S_{\nu(\theta_0)} \) is smooth near \( p_i \). Thus, by the above paragraph, it follows that if \( |\theta - \theta_0| < \epsilon_1 \), for some sufficiently small \( \epsilon_1 > 0 \), then \( T \) meets \( \partial S_{\nu(\theta)} \) transversely as well. Then it follows that for some neighborhood \( W \) of \( \{ p_1, p_2, p_3 \} \), \( (T - \{ p_1, p_2, p_3 \}) \cap W \subset S_{\nu(\theta)} \) for all \( \theta \) such that \( |\theta - \theta_0| < \epsilon_1 \).

Next note that \( T - W \) is compact, and the mapping \( \theta \mapsto \sigma_{\nu(\theta)} \) is continuous; therefore, since by assumption \( \sigma_{\nu(\theta_0)} > 0 \) on \( T - W \), it follows that there exists an
$\epsilon_2 > 0$ such that $\sigma_{v(\theta)} > 0$ on $T - W$ for all $\theta$ such that $|\theta - \theta_0| < \epsilon_2$. This yields that $T - W \subset S_{v(\theta)}$ for all $\theta$ such that $|\theta - \theta_0| < \epsilon_2$.

From the previous two paragraphs it follows, setting $\epsilon := \min\{\epsilon_1, \epsilon_2\}$, we have $(T - \{p_1, p_2, p_3\}) \subset S_{v(\theta)}$ for all $\theta$ such that $|\theta - \theta_0| < \epsilon$, which completes the proof. \hfill $\square$

**Note 5.4.** Theorem 1.1 does not remain valid if the shadows are defined as the sets where $\langle n(p), u \rangle \geq 0$. For instance, the standard torus of revolution would be a counterexample.

**Note 5.5.** Theorem 1.1 does not remain valid without the compactness assumption; the hyperbolic paraboloid given by the graph of $z = xy$ would be a counterexample. This follows because here the unit normal vectorfield $n$ is a homeomorphism into a hemisphere. Thus the preimage of any open hemisphere under $n$ is simply connected.

### 6. Proof of Theorem 1.2

**Definition 6.1.** We say an immersion $\gamma : S^1 \cong \mathbb{R}/2\pi \to \mathbb{R}^3$ is a skew loop if it has no pair of distinct parallel tangent lines, i.e.,

$$\gamma'(t) \times \gamma'(s) \neq 0$$

for all $t, s \in [0, 2\pi)$, $t \neq s$.

A specific example of a skew loop, formulated by Ralph Howard, is as follows:

**Example 6.2.** Let $\gamma(t) := (x(t), y(t), z(t))$, where

$$
x(t) := -\cos(t) - \frac{1}{20} \cos(4t) + \frac{1}{10} \cos(2t),
$$

$$
y(t) := +\sin(t) + \frac{1}{10} \sin(2t) + \frac{1}{20} \sin(4t),
$$

$$
z(t) := -\frac{46}{75} \sin(3t) - \frac{2}{15} \cos(3t) \sin(3t),$$

and $t \in [0, 2\pi]$. A computation of the tangential indicatrix $T(t) := \gamma'(t)/\|\gamma'(t)\|$ shows that $T(t) \neq \pm T(s)$ for all $t, s \in [0, 2\pi)$, $t \neq s$. Thus $\gamma$ is a skew loop. Figure 1 shows the pictures of a tube built around $\gamma(S^1)$.

If $\gamma : S^1 \to \mathbb{R}^3$ is an immersion, then the unit normal bundle of $\gamma$ consists of all pairs $(p, \nu) \in S^1 \times S^2$ such that $\langle \gamma'(p), \nu \rangle = 0$. Since this bundle is homeomorphic to a torus, the following proposition yields Theorem 1.2.

**Proposition 6.3.** Let $\gamma : S^1 \to \mathbb{R}^3$ be a skew loop and $M$ be the unit normal bundle of $\gamma$. For $\epsilon > 0$, define $f_\epsilon : M \to \mathbb{R}^3$ by

$$f_\epsilon(p, \nu) := \gamma(p) + \epsilon \nu.$$

Then, for $\epsilon$ sufficiently small, $f_\epsilon$ is a smooth immersion, and for all $u \in S^2$, $S_u$ is connected. If $\gamma$ is an embedding, then $f_\epsilon$ is an embedding as well.
Proof. That $f_c$ is a smooth immersion and is an embedding when $\gamma$ is embedded follows from the tubular neighborhood theorem. Let $n: M \to \mathbb{S}^2$ be the unit normal vector field given by $n(p, \nu) = \nu$, and $\pi: M \to \mathbb{S}^1$ be given by $\pi(p, \nu) = p$. For every $p \in \mathbb{S}^1$, let $F_p := \pi^{-1}(p)$ be the corresponding fiber. Note that $n$ embeds $F_p$ into the great circle in $\mathbb{S}^2$ which lies in the plane perpendicular to $T(p)$. Further recall that $S_u = n^{-1}(\mathbb{S}^2_u)$ where $\mathbb{S}^2_u$ is the open hemisphere determined by $u$. Thus there are only two possibilities for each $p \in \mathbb{S}^1$: either $F_p$ intersects $S_u$ in an open half-circle, or $F_p$ is disjoint from $S_u$. The latter occurs if and only if $T(p)$ is parallel to $u$, which, since $\gamma$ is skew, can occur at most once. Hence, it follows that $S_u$ is either homeomorphic to a disk or an annulus. In particular, $S_u$ is connected for every $u \in \mathbb{S}^2$. \hfill $\square$

**Question 6.4.** Let $M$ be a closed oriented 2-dimensional manifold with topological genus $g(M) \geq 2$. Does there exist an embedding, or an immersion, $f: M \to \mathbb{R}^3$ such that $S_u$ is connected for all $u \in \mathbb{S}^2$?

**Note 6.5.** Skew loops were first discovered by B. Segre [16] to disprove a conjecture of H. Steinhaus (see also [15]). More recently, it has been shown that there exists a skew loop in each knot class [18], and every pair of knots may be realized with the same tangential indicatrix [1].

**Note 6.6.** A general procedure for constructing skew loops is as follows. Let $T \subset \mathbb{S}^2$ be a smooth simple closed curve such that (i) the origin is contained in the interior of the convex hull of $T$, $(0, 0, 0) \in \text{int conv } T$, and (ii) $T$ does not contain any pair of antipodal points, $T \cap -T = \emptyset$. Figure 2 shows an example. Let $T(s), s \in \mathbb{R}$, denote a periodic parameterization of $T$ by arclength. So, assuming $T$ has total length $L$, we have $T(s + L) = T(s)$. Since $(0, 0, 0) \in \text{int conv } T$, there exists a function $\rho(s)$ with period $L$ such that $\int_0^L \rho(s)T(s) \, ds = 0$ [10, p. 168]. Set

$$\gamma(t) := \int_0^t \rho(s)T(s) \, ds.$$
Then $\gamma(t + L) = \gamma(t)$. Further, $\gamma'(t)/\|\gamma'(t)\| = T(t)$. Thus $\gamma$ is a closed curve whose tangential spherical image coincides with $T$. Hence $\gamma$ is a skew loop.

**Note 6.7.** With the sole exception of ellipsoids, every closed surface immersed in $\mathbb{R}^3$ admits a skew loop [8].

7. **Proof of Theorem 1.3**

We follow a modified outline of the proof of Theorem 1.1, which again proceeds by contradiction. Suppose that $M$ is homeomorphic to $S^2$ and $S_u$ is connected for all $u \in S^2$. If $f$ is not a convex embedding, let $u_0$ and $v_0$ be as in Lemma 5.1, and $v(\theta)$ be as defined by (1).

**Definition 7.1.** The augmented shadow $\tilde{S}_{v(\theta)}$ is the union of $S_{v(\theta)}$ with all components $X$ of $H_{v(\theta)}$ such that $U - X \subset S_{v(\theta)}$ for an open neighborhood $U$ of $X$.

Then $\tilde{S}_{v(\theta)}$ satisfies the conditions of the following lemma:

**Lemma 7.2.** If $U \subset S^2$ is a connected open set, and $S^2 - U$ is also connected and has an interior point, then $U$ is simply connected.

*Proof.* Let $p$ be an interior point of $S^2 - U$. Then the stereographic projection maps $U$ into a connected open set with connected complement. Thus, by [9, Thm. 11.4.1], $U$ is simply connected. \( \square \)

So $\tilde{S}_{v(\theta)}$ is simply connected. Further:

**Lemma 7.3.** For all $\theta \in \mathbb{R}$, $\tilde{S}_{v(\theta)}$ is a domain adjacent to and regular near $p_i$.

*Proof.* This follows just as in the proof of Lemma 5.2, once we observe that whenever $\partial S_{v(\theta)} = H_{v(\theta)} = \partial S_{-v(\theta)}$ is regular in some open neighborhood, then $\partial \tilde{S}_{v(\theta)}$, and $\partial S_{v(\theta)}$ coincide within that neighborhood. \( \square \)

Thus each $\theta$ induces a permutation $\tilde{\alpha}_\theta := \alpha_{(S_{v(\theta)})} \sim \{p_1, p_2, p_3\}$ which satisfies the enumerated properties in Proposition 4.4. In particular $\tilde{\alpha}_0 \neq \tilde{\alpha}_{\pi_1}$, because since $\partial S_{v(0)} = \partial (S_{-v(0)})$ is by Lemma 5.1 a regular curve, it follows that $\partial \tilde{S}_{v(0)} = \partial \tilde{S}_{-v(0)}$ is a regular curve as well. So it remains to verify the following lemma which shows
that $\theta \mapsto \tilde{\alpha}_\theta$ is locally constant. This would yield that $\tilde{\alpha}_0 = \tilde{\alpha}_\pi$ which is the desired contradiction.

**Lemma 7.4.** For each $\theta_0 \in \mathbb{R}$ there exists an $\epsilon > 0$ such that if $|\theta - \theta_0| < \epsilon$ then $\tilde{S}_{v(\theta)}$ and $\tilde{S}_{v(\theta_0)}$ have a common triangle (with vertices at $\{p_1, p_2, p_3\}$).

**Proof.** This is an immediate consequence of Lemma 5.3 where it was proved that $S_{v(\theta)}$ and $S_{v(\theta_0)}$ have a triangle in common (the proof of Lemma 5.3 makes no use of the simply connectedness assumption on $S_{v(\theta)}$).

\[ \square \]

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**References**


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