Algorithmic Stochastic Localization for the Sherrington-Kirkpatrick model

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Joint work with Andrea Montanari & Mark Sellke

The Sherrington-Kirkpatrick model

The Boltzmann-Gibbs measure:

$$\mu(\sigma) \propto e^{-\langle \sigma, M\sigma \rangle/2}, \quad \sigma \in \{\pm 1\}^n.$$

$M = (M_{ij})_{i,j=1}^n$: The interaction matrix

$$M = M^\top \qquad M_{ij} \sim N(0, \beta^2/n) \quad i < j$$

β : The inverse temperature

μ favors vectors (configurations) with low energy $\langle \sigma, M\sigma \rangle$

A phase transition

High temperature (paramagnetic) phase: $\beta \leq 1$

$$\sigma^1, \sigma^2 \sim \mu \qquad\qquad \frac{1}{n}\langle \sigma^1, \sigma^2 \rangle \longrightarrow 0$$

The model as “simple” characteristics

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Low temperature (spin glass) phase: $\beta > 1$

$\frac{1}{n}\langle \sigma^1, \sigma^2 \rangle$ converges to a non-trivial random variable

Highly complex structure. Sophisticated mathematical description.

Main question

Can we approximately sample from the SK measure in polynomial time?

Folklore belief:

1. Sampling should be easy in the high temperature phase
2. Correct answer is unclear for low temperature

Glauber dynamics

1. $\sigma^0 \sim \mathrm{Unif}(\{\pm 1\}^n)$.
2. At time t: $i \sim \mathrm{Unif}(\{1, \cdots, n\})$.
3. Sample $\varepsilon \sim \mu(\cdot | (\sigma_j^t)_{j \neq i})$.
4. Set $\sigma_i^{t+1} = \varepsilon$, $\sigma_{\sim i}^{t+1} = \sigma_{\sim i}^t$.

$$\mu(\sigma_i | \sigma_{\sim i}) \propto e^{-\sigma_i (\sum_{j \neq i} M_{ij} \sigma_j)/2}.$$

Does this mix in polynomial time?

Mixing time & Poincaré inequality

We say that μ satisfies a **Poincaré inequality (PI)** if

For all $f : \{-1, +1\}^n \to \mathbb{R}$ $\qquad \mathrm{Var}_\mu(f) \le \frac{1}{\gamma}\mathcal{E}_\mu(f, f) \qquad$ for some $\gamma > 0$

$$\mathrm{Var}_\mu(f) = \mathbb{E}_\mu\left[(f(\sigma) - \mathbb{E}_\mu[f(\sigma)])^2\right]$$

Variance

$$\mathcal{E}_\mu(f, f) = \mathbb{E}_\mu \sum_{i=1}^{n} \big(\mathbb{E}_\mu[f(\sigma)|\sigma_{\sim i}] - f(\sigma)\big)^2$$

Dirichlet form

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$$\mathcal{E}_\mu(f,f) = \mathbb{E}_\mu \sum_{i=1}^{n} \left(\mathbb{E}_\mu[f(\sigma)|\sigma_{\sim i}] - f(\sigma)\right)^2$$ Dirichlet form

Lemma: If μ satisfies PI with constant γ then Glauber dynamics mixes after

$t_{\mathrm{mix}} = O(n/\gamma)$ steps.

Mixing time & Poincaré inequality

Theorem: μ satisfies PI with constant $\gamma = 1 - \Delta$

[Eldan-Koehler-Zeitouni 2020]

$$\Delta = \lambda_{\max}(M) - \lambda_{\min}(M) = 4\beta + o_n(1)$$

Proof: Reduction to rank-one model using *Stochastic localization*

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Conjecture: Linear-time mixing for all $\beta < 1$

Decomposition into a mixture of products

[Bauerschmidt-Beaudineau 2019]

Theorem:

$$\mu = \int \mu_\tau m(d\tau) \qquad \mu_\tau(\sigma) \propto e^{\langle \tau, \sigma \rangle}$$

m is log-concave for all $\beta < 1/4$

Decomposition into a mixture of products

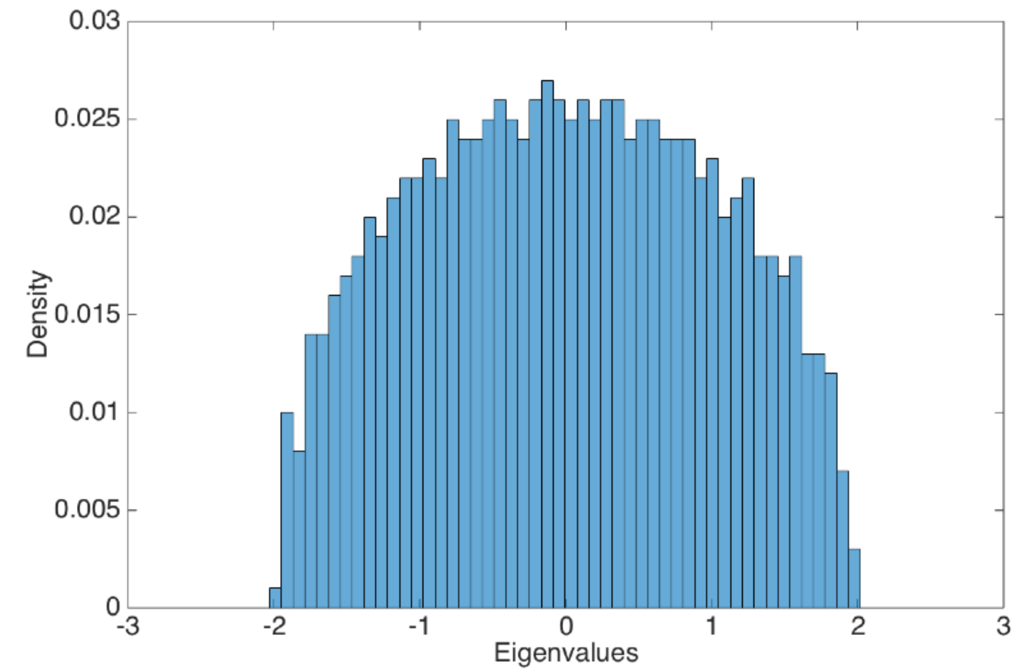
[Bauerschmidt-Beaudineau 2019]

Since $\sigma \in \{-1,+1\}^n$ we can add a diagonal term to M without affecting μ

$$M \longrightarrow M + \delta I$$

so that $$0 \preceq M \preceq cI$$

$$c = (4\beta + o_n(1))$$



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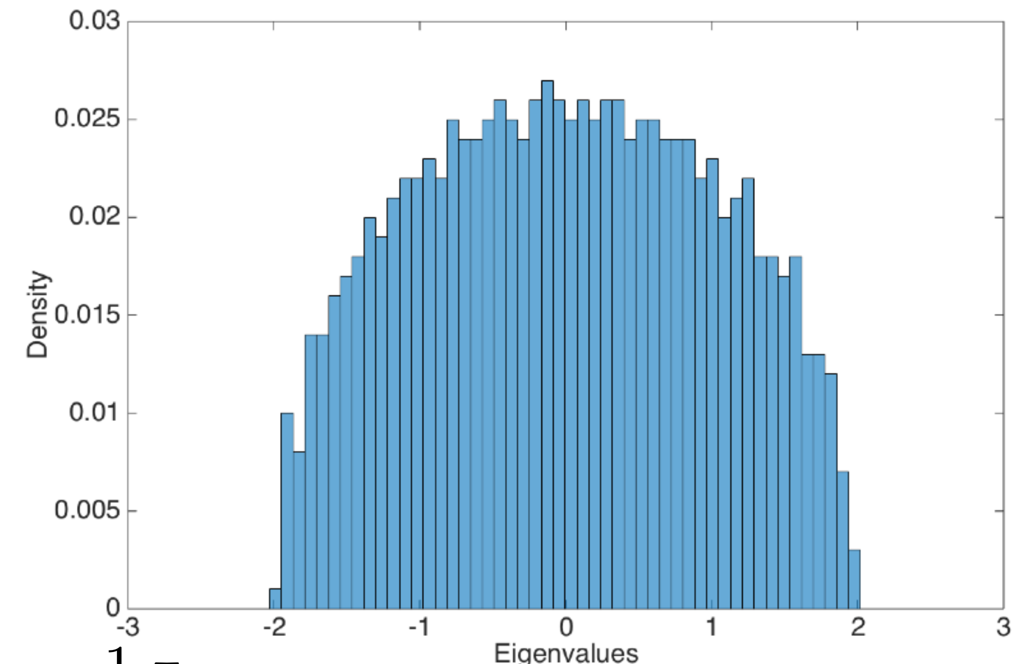

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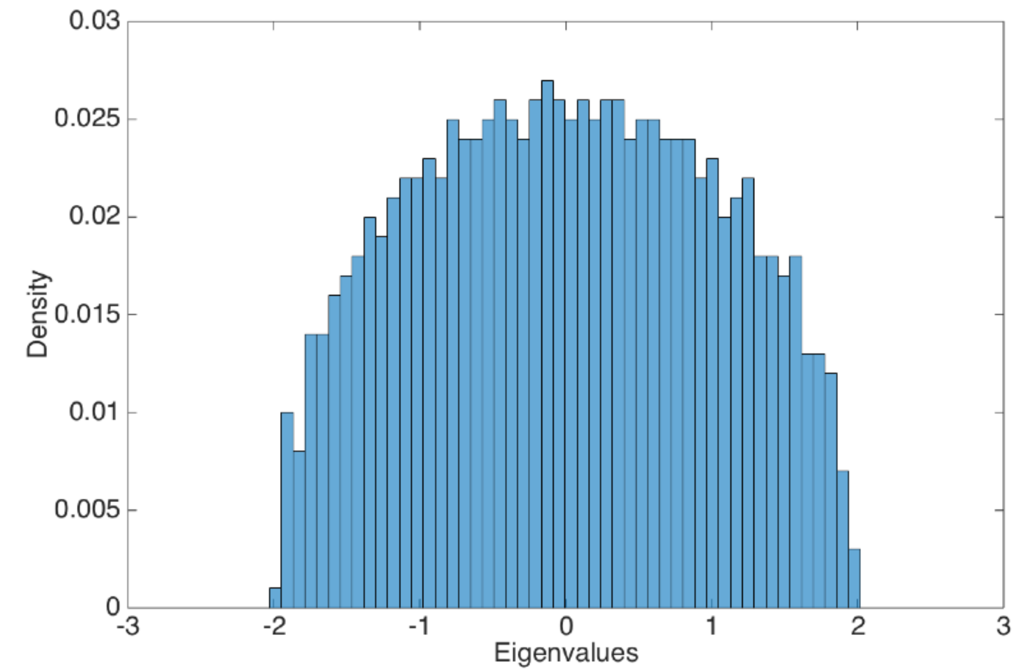

There exists $B \succeq 0$ such that $M^{-1} = B^{-1} + c^{-1} I$

$$\implies \qquad e^{-\langle \sigma, M\sigma\rangle/2} = C \int e^{-c\|\varphi - \sigma\|^2/2} e^{-\langle \varphi, B\varphi\rangle/2} d\varphi.$$

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$$e^{-\langle \sigma, M\sigma \rangle/2} = C \int e^{-c\|\varphi-\sigma\|^2/2} e^{-\langle \varphi, B\varphi \rangle/2} d\varphi.$$

1. Construct a joint distribution

$$\pi(d\sigma, d\varphi) \propto e^{-c\|\varphi-\sigma\|^2/2-\langle \varphi, B\varphi \rangle/2} \mu_0(d\sigma) d\varphi$$

with marginal μ on σ.

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3. The marginal on φ is $\nu(d\varphi) \propto e^{-\langle \varphi, (B+cI)\varphi \rangle/2 + \sum_{i=1}^n V(\varphi_i)} d\varphi$,

where $V(x) = \log\cosh(cx)$.

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$\text{Hess} \succeq cI$ $\quad$ $\text{Hess} \preceq c^2 I$

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$\Longrightarrow$ ν is **log-concave** if $c^2 \leq c$ i.e., if $\beta < 1/4$

The algorithm

1. Use Langevin dynamics to sample φ from ν (mixes in linear time).

2. Sample $\sigma \sim \pi(\cdot|\varphi)$

$$\pi(d\sigma|\varphi) \propto e^{c\langle\varphi,\sigma\rangle}\mu_0(d\sigma).$$

$$\pi(\sigma_i = 1|\varphi) = \frac{e^{c\varphi_i}}{e^{c\varphi_i} + e^{-c\varphi_i}}$$

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The method relies on a clever decomposition. Works up to $\beta < 1/4$

Question: Is it possible to decompose μ into a mixture of tilts

$$\mu = \int \mu_\tau m(d\tau) \qquad \mu_\tau(\sigma) \propto e^{\langle\tau,\sigma\rangle}$$

where it is <u>easy to sample</u> from m for all $\beta < 1$?

Perhaps recursively?

Main result

Theorem 2.1. *For $\varepsilon > 0$ and $\beta < \frac{1}{2}$ there exists a polynomial-time randomized algorithm which takes $(\beta, \boldsymbol{A})$ as input and outputs a random point $\boldsymbol{x}^{\mathrm{alg}} \in \{-1, +1\}^n$ with law $\mu_{\boldsymbol{A}}^{\mathrm{alg}}$ such that with probability $1 - o_n(1)$ over $\boldsymbol{A}$,*

$$W_{2,n}(\mu_{\boldsymbol{A}}^{\mathrm{alg}}, \mu_{\boldsymbol{A}}) \leq \varepsilon\,. \tag{2.10}$$

Runtime: $\mathrm{poly}(n, 1/\epsilon)$

$$W_{2,n}(\mu, \nu)^2 = \inf_{\pi \in \mathcal{C}(\mu,\nu)} \frac{1}{n} \mathbb{E}_\pi \left[\left\| \boldsymbol{X} - \boldsymbol{Y} \right\|_2^2 \right],$$

Result relies of a discretization of the stochastic localization process

Stochastic Localization

Stochastic localization [Eldan 2013, 2018]

Fix a measure μ on $\mathbb{R}^n$

Construct a measure-valued process $(\mu_t)_{t\geq 0}$ as follows:

$$L_t(x) = \frac{\mathrm{d}\mu_t}{\mathrm{d}\mu}(x) \qquad L_0 = 1$$

$$\mathrm{d}L_t(x) = L_t(x)\,\langle x - m_t, \mathrm{d}B_t\rangle \qquad \forall t \geq 0,\ \forall x \in \mathbb{R}^n$$

$$m_t = \int x\mu_t(\mathrm{d}x)$$

$(B_t)_{t\geq 0}$ Brownian motion

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Strong solution exists under mild assumptions

$(\mu_t)_{t\geq 0}$: stochastic localization process

Stochastic localization [Eldan 2013, 2018]

Properties:

1. $(L_t)_{t\geq 0}$, $(\mu_t)_{t\geq 0}$ and $(m_t)_{t\geq 0}$ are martingales

   In particular $\mu = \mathbb{E}\mu_t$

2. $\forall t \geq 0$ $\qquad \mathbb{E}\mathrm{Cov}(\mu_t) \preceq \frac{1}{t} I$

3. Consequence of 1 and 2:

$$m_t \xrightarrow[t\to\infty]{\mathrm{d}} m_\infty \sim \mu$$

An equivalent formulation

Exponential tilts: For any $y \in \mathbb{R}^n$ define the measure

$$\mu_{t,y}(\mathrm{d}x) = \frac{1}{Z(t,y)} e^{\langle y,x\rangle - t\|x\|^2/2} \mu(\mathrm{d}x)$$

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$$\mathrm{d}y_t = m(t, y_t)\mathrm{d}t + \mathrm{d}B_t, \quad y_0 = 0\,.$$

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Lemma: $$(\mu_{t,y_t})_{t\ge 0} \stackrel{\mathrm{d}}{=} (\mu_t)_{t\ge 0}$$

[Eldan, Shamir 2020]

Discretized SL

For $k = 0, 1, 2, \cdots$

1. Given an external field y_ℓ compute the mean vector

$$m_\ell \simeq m(y_\ell) = \int x \mu_{y_\ell}(\mathrm{d}x)$$

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Hope that the discretized iteration converges to the continuum SDE…

Conditions

We need to compute approximations $\widehat{m}(y)$ of the mean vector $m(y)$

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which are sufficiently accurate and regular:

1. **Approximation**: $$\frac{1}{n}\|\widehat{m}(y) - m(y)\|^2 = o_n(1)$$

2. **Regularity:**

$y \mapsto \widehat{m}(y)$ Lipschitz *uniformly in the approximation error*

Then output $\widehat{m}_L$ for $L = T/\delta$ $\delta \to 0,\ T \to \infty$

A remark: Semi log-concavity

Log-Laplace transform: $$\mathcal{L}[\nu](x) = \log \int_{\mathcal{C}_n} e^{\langle x,y\rangle} d\nu(y), \quad \forall x \in \mathbb{R}^n.$$

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[Eldan, Shamir 2020]

Definition 1. (Semi log-concave measures). Given a measure ν on $\mathcal{C}_n$, We say that ν is β-semi-log-concave if

$$\nabla^2 \mathcal{L}[\nu](x) \preceq \beta \mathbf{I}_n, \quad \forall x \in \mathbb{R}^n, \tag{2}$$

where the inequality is in the positive-definite sense.

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Conjecture [Talagrand]: The SK measure is C-semi-log-concave for all $\beta < 1$

Confirmed by Eldan-Koehler-Zeitouni 2020 for all $\beta < 1/4$

Computing the means

Approximate message passing: Standard technology for computing $m(y_k)$

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Approximate message passing: Standard technology for computing $m(y_k)$

Two issues technical issues (in the analysis):

1. y_k depends on A (we would rather have them be independent!)

   Solved by introducing a *planted* model

2. The Lipschitz constant of AMP after k iterations blows up with k

   Solved by modifying the algorithm

   Cause of the bottleneck $\beta < 1/2$

Another characterization of SL

1. Sample $x_0 \sim \mu$

2. Let $y_t = t x_0 + B_t$

3. Look at $\mu_t = \mathrm{Law}(x_0 \,|\, (y_s)_{s \leq t})$

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Lemma: $(\mu_t)_{t \ge 0} \stackrel{\mathrm{d}}{=} \text{SL process}$

‘Planted’ and ‘random’ models

Random model

$$
\begin{aligned}
A &\sim \mathrm{GOE}(n) \\
\mathrm{d}y_t &= m(y_t)\mathrm{d}t + \mathrm{d}B_t \\
x &\sim \mu_{A,y_t} \propto e^{(\beta/2)\langle x, Ax\rangle + \langle y_t, x\rangle} .
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Planted model

$$x_0 \sim \mathrm{Unif}(\{-1,+1\}^n)$$
$$A = \beta x_0 x_0^\top + W,\; W \sim \mathrm{GOE}(n)$$
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$$\mathbb{P} = \mathrm{Law}(A, y)$$

Lemma: $\mathbb{P}$ and $\mathbb{Q}$ are mutually contiguous for all $\beta < 1$

We can conduct the analysis on the planted model instead !

Stability and Chaos

Let $$A_s = \sqrt{1 - s^2} A + s A'$$

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Thm[stability]: Our algorithm is stable in the following sense:

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Thm[chaos]: For all $\beta > 1$ $$\inf_{s\in(0,1)} \liminf_{n\to\infty} W_{2,n}\big(\mu_A, \mu_{A_s}\big) > 0$$

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For all $\beta > 0$ $$\lim_{s\to 0}\lim_{n\to\infty} W_{2,n}(\mu_A^{\rm alg}, \mu_{A_s}^{\rm alg}) = 0$$

$\Longrightarrow$ The same is true for the SK measure μ_A for all $\beta < 1/2$

Thm[chaos]: For all $\beta > 1$ $$\inf_{s\in(0,1)}\liminf_{n\to\infty} W_{2,n}\big(\mu_A, \mu_{A_s}\big) > 0$$

$\Longrightarrow$ No stable algorithm can approximate the SK measure at low temperature