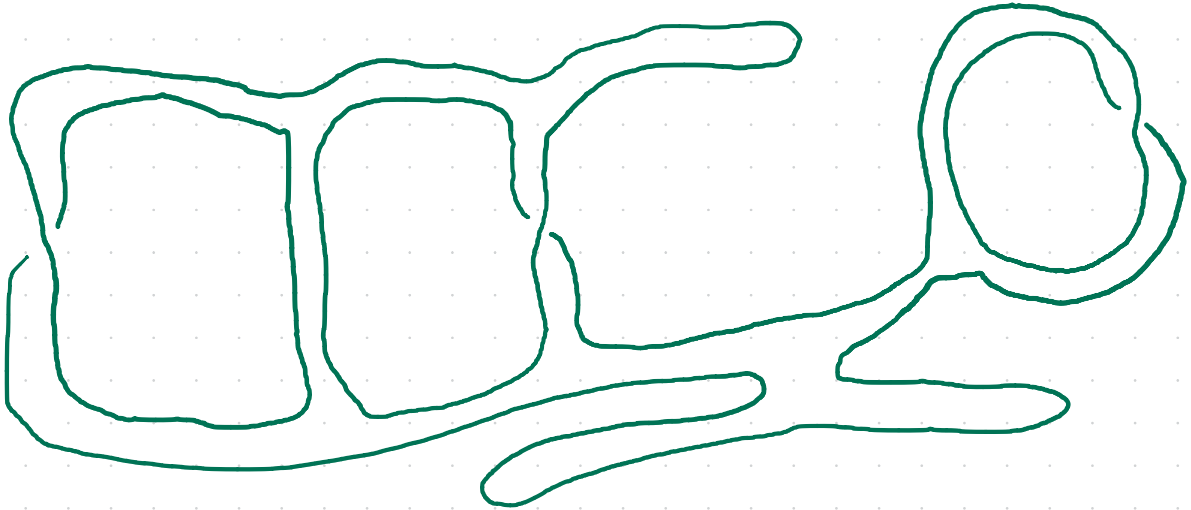
Unintuitive properties of Urysohn 1-width

Hannah Alpert, Auburn University

Joint work with Panos Papasoglu, Arka Banerjee, Alexey Balitskiy, and Larry Guth

$UW_1(X) < \varepsilon$ means $\exists f : X \to$ graph continuous such that $f(x_1) = f(x_2) \Rightarrow d(x_1, x_2) < \varepsilon$.

Example: surface within $\frac{\varepsilon}{10}$ of its boundary



Main open question:

$$UW_1(\tilde{X}) \leq 1 \overset{?}{\Longrightarrow} UW_1(X) \leq \text{const}?$$

<u>Thm</u> If X is a surface with a Riemannian metric, then

$$UW_1(X) \leq UW_1(\tilde{X}).$$

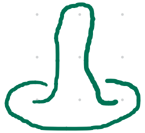
Fact In $\mathbb{R}^2$,

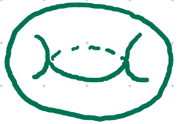
$$UW_1(r\text{-disk}) = r \cdot UW_1(1\text{-disk}) > 0.$$

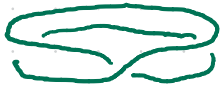
$$UW_1(X) \text{ small} \overset{\checkmark}{\Longrightarrow} X \text{ contains no large disk}$$

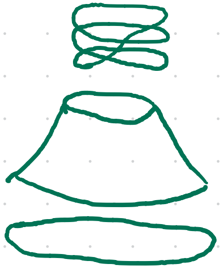
$$\underset{\text{not literally}}{\Longleftarrow}$$

A large disk still has large UW_1 if you cut out a small disk and replace by . . .

→ a long finger 

→ a handle 

→ a Möbius band 

→ $S^1 \xrightarrow{\times k} S^1$ mapping cylinder 

Gromov-Lawson '83: If M^3 has scalar curvature ≥ 1, then

every null-homologous loop is fillable by a surface in its 2π-neighborhood.

"M has loop-filling within 2π"

$$\overset{???}{\Longrightarrow} \quad UW_1(M) \leq \text{const}?$$

Q (Gromov): If X has loop-filling within r, must $UW_1(X) \leq \text{const} \cdot r$?

A (Balitskiy–Berdnikov): Not with $\mathbb{Z}$-coefficients, maybe with $\mathbb{Q}/\mathbb{Z}$.

Example: large disk interrupted by many $S^1 \xrightarrow{\times k} S^1$ mapping cylinders

Gromov-Lawson '83: If X has loop-filling within r, and $H_1(X;\mathbb{Q})=0$, then $UW_1(X)\leq 6r$.

So,

M^3 has PSC $\geq 1 \Rightarrow UW_1(\tilde{M})\leq$ const.

Main open question:

$$\mathrm{uw}_1(X) \overset{?}{\leq} \mathrm{const} \cdot \mathrm{uw}_1(\tilde{X}) \; ?$$

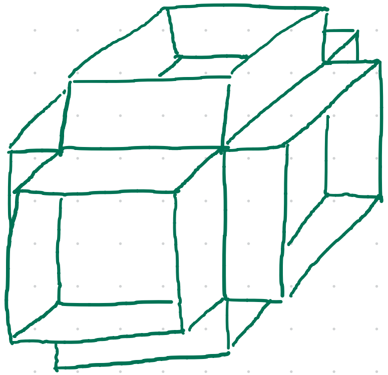
	true	false
known	X surface $\dim H_1(X;\mathbb{Q}) \leq \mathrm{const}$	???
expected	X closed 3-mfld	X arbitrary 2-complex X high-dim manifold

Caveats:

(1) $\exists$ closed surface Σ and covering space $\hat{\Sigma}$ with $UW_1(\Sigma)$ large, $UW_1(\hat{\Sigma})$ small.

(2) $\exists$ closed 4-mfld M with $UW_2(M)$ large, $UW_2(\tilde{M})$ small.

$\hat{\Sigma}$ = surface in $\mathbb{R}^3$ equidistant between (1-skeleton of $\mathbb{Z}^3$) and (1-skeleton of $\mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$)



separates congruent regions R_1, R_2 and both have small UW_1

Σ = quotient of $\hat{\Sigma}$ by 3 translations

$$(1000, 0, 0)$$
$$(0, 1000, 0)$$
$$\left(\tfrac{1}{2}, \tfrac{1}{2}, 1000+\tfrac{1}{2}\right) \leftarrow \text{swaps } R_1 \text{ and } R_2$$

This has large UW_1.

Conjectures in geometric analysis:

If M^n has PSC ≥ 1, then...

→ $UW_{n-2}(M) \leq \text{const}(n)$

$$\Downarrow$$

→ $UW_{n-1}(M) \leq \text{const}(n)$

$$\Downarrow$$

→ $UW_{n-1}(\widetilde{M}) < \infty$

$$\Downarrow$$

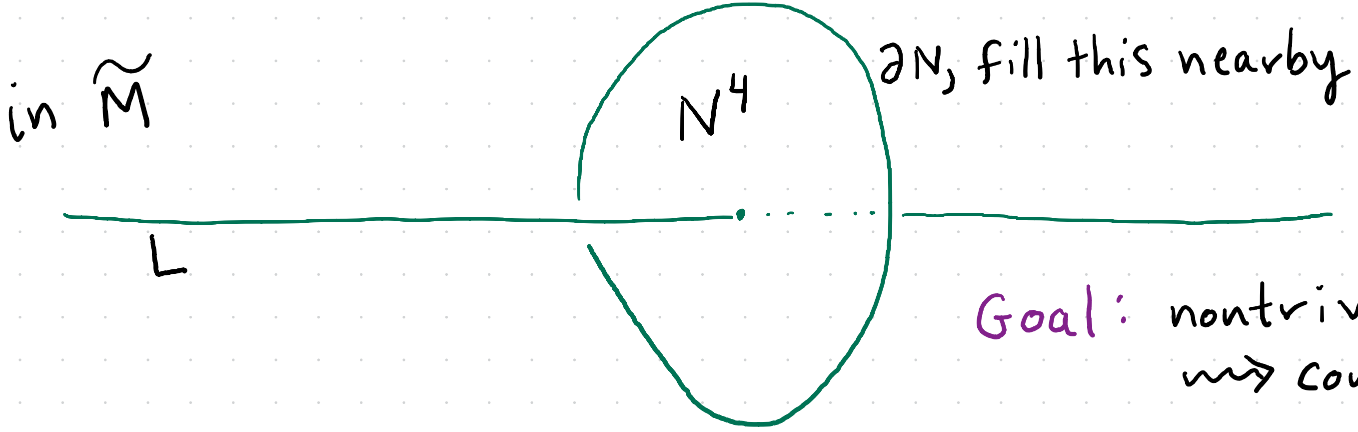
→ $\widetilde{M}$ is not contractible

Chodosh–Li '23:

M^n closed, PSC, $n \leq 5$ $\Rightarrow$ $\tilde{M}$ is not contractible

μ-bubble technique: can construct near-PSC near-spherical hypersurfaces in near-PSC manifolds

Base case: 2 dimensions



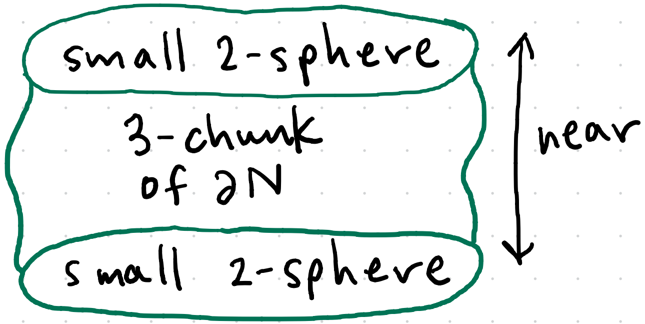
Goal: nontrivial 4-cycle
$\rightsquigarrow$ contradiction

- construct N (4-dim)
- construct ∂N (3-dim)
- construct small 2-spheres cutting ∂N into bounded chunks
- fill 2-spheres, then chunks of ∂N

∂N has small UW_1

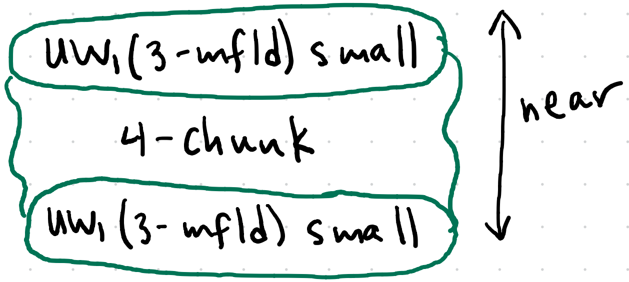
Why doesn't it work for $n=6$?

n=5 case:



$\Rightarrow$ 3-chunk has small diameter

n=6 case:



$\not\Rightarrow$ 4-chunk has small UW_1

Hopes and dreams:

→ When does UW_1 behave well?

→ When it doesn't, should we define something else that's better?

One more example:

A metric on the Θ-graph where every loop can map with small fibers to a tree, but the whole graph cannot