Stability of the Prekopa-Leindler inequality and the unconditional Logarithmic Brunn-Minkowski Inequality

Károly Böröczky
Alfréd Rényi Institute of Mathematics

joint with Apratim De, Alessio Figalli, Joao P. Ramos

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Sketch of the talk

Part 1 Stability of the Prekopa-Leindler inequality
  ▶ for log-concave functions - detailed discussion
  ▶ general case - announcement

Part 2 Stability of the Bollobas-Leader inequality for coordinatewise product of unconditional bodies

Part 3 Stability of the Log-Minkowski inequality

\[ \int_{S^{n-1}} \log \frac{h_C}{h_K} \frac{dV_K}{V(K)} \geq \frac{1}{n} \cdot \log \frac{V(C)}{V(K)} \]

with hyperplane symmetries where \( dV_K = \frac{1}{n} h_K dS_K \) is the cone-volume measure (including stability of the Logarithmic Brunn-Minkowski inequality with hyperplane symmetries)
Prekopa-Leindler inequality

\[ f : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \text{ log-concave} \]
\[ \iff f((1 - \alpha)x + \alpha y) \geq f(x)^{1-\alpha}f(y)^{\alpha} \quad \forall x, y \in \mathbb{R}^n, \forall \alpha \in (0, 1) \]
\[ \iff f = e^{-\varphi} \text{ for convex } \varphi : \mathbb{R} \to (-\infty, \infty] \]

Theorem (Prekopa, Leindler, Dubuc)

If \( \lambda \in (0, 1) \) and \( h, f, g : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) satisfy \( \circ < \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g < \infty \),

\[ h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda}g(y)^{\lambda} \quad \text{for } x, y \in \mathbb{R}^n, \text{ then} \]

\[ \int_{\mathbb{R}^n} h \geq \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \cdot \left( \int_{\mathbb{R}^n} g \right)^{\lambda}, \]

equality implying that for \( a = \int_{\mathbb{R}^n} g / \int_{\mathbb{R}^n} f \), there exist \( w \in \mathbb{R}^n \) and log-concave function \( \tilde{h} \) such that \( h = \tilde{h}, \; f(x) = a^{-\lambda}\tilde{h}(x - \lambda w), \)
\( g(y) = a^{1-\lambda}\tilde{h}(y + (1 - \lambda)w) \) almost everywhere.

Remark \( \sup_z=(1-\lambda)x+\lambda y \; f(x)^{1-\lambda}g(y)^{\lambda} \) may not be measurable, but it is log-concave, if \( f, g \) log-concave.
Brunn-Minkowski inequality and its stability

$K, C$ convex bodies in $\mathbb{R}^n$

Brunn-Minkowski inequality $\lambda \in (0, 1)$

$$V((1 - \lambda) K + \lambda C) \geq V(K)^{1-\lambda} V(C)^{\lambda}.$$  

$\alpha = V(K)^{-\frac{1}{n}}, \beta = V(C)^{-\frac{1}{n}}, \sigma = \max \left\{ \frac{V(C)}{V(K)}, \frac{V(K)}{V(C)} \right\}$

$$A(K, C) = \min_{x \in \mathbb{R}^n} V \left( (\alpha K) \Delta (x + \beta C) \right)$$

Theorem [Figalli, Maggi, Pratelli $\sim$ 2010]

$$V \left( \frac{1}{2} K + \frac{1}{2} C \right) \geq \sqrt{V(K) \cdot V(C)} \left[ 1 + \frac{(\sigma - 1)^2}{32n\sigma^2} + \frac{\gamma(n)}{\sigma^{\frac{1}{n}}} \cdot A(K, C)^2 \right].$$

Remark  First stability version by Minkowski
Stability of Prekopa-Leindler for log-concave functions

Theorem (B., De)

If \( \tau \in (0, \frac{1}{2}] \), \( \lambda \in [\tau, 1 - \tau] \) and \( f, g \) are log-concave on \( \mathbb{R}^n \) satisfying

\[
\int \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda \, dz \leq (1 + \varepsilon) \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g \right)^\lambda
\]

for \( \varepsilon > 0 \), then for \( a = \int_{\mathbb{R}^n} f / \int_{\mathbb{R}^n} g \), there exists \( w \in \mathbb{R}^n \) such that

\[
\int_{\mathbb{R}^n} |f(x) - a \cdot g(x + w)| \, dx \leq c^n n^n \left( \frac{\varepsilon}{\tau} \right)^{\frac{1}{19}} \int_{\mathbb{R}^n} f.
\]
Stability of Prekopa-Leindler - announcement

Theorem (B., Figalli, Ramos)

For $\tau \in (0, \frac{1}{2}]$ and $\lambda \in [\tau, 1 - \tau]$, if $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are measurable functions such that $h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda}g(y)\lambda$ holds for $x, y \in \mathbb{R}^n$, and

$$\int_{\mathbb{R}^n} h \leq (1 + \varepsilon) \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g \right)^\lambda,$$

then setting $a = \int_{\mathbb{R}^n} g / \int_{\mathbb{R}^n} f$, $\exists$ log-concave $\tilde{h}$ and $w \in \mathbb{R}^n$ with

$$\int_{\mathbb{R}^n} |a^\lambda f - \tilde{h}(\cdot + \lambda w)| < M_n \frac{\varepsilon Q_n(\tau)}{\tau N_n} \int_{\mathbb{R}^n} h,$$

$$\int_{\mathbb{R}^n} |a^{\lambda-1} g - \tilde{h}(\cdot + (\lambda - 1)w)| < M_n \frac{\varepsilon Q_n(\tau)}{\tau N_n} \int_{\mathbb{R}^n} h,$$

$$\int_{\mathbb{R}^n} |h - \tilde{h}| < M_n \frac{\varepsilon Q_n(\tau)}{\tau N_n} \int_{\mathbb{R}^n} h.$$
Very little history

- Stability of Brunn-Minkowski w.r.t. Hausdorff distance: Diskant, Groemer (Isoper. Inequality: Schneider, Fuglede)
- Optimal stability of Isoperimetric Inequality w.r.t. symmetric difference metric: Fusco, Maggi, Pratelli \( \sim 2008 \)
- Optimal stability of Brunn-Minkowski for convex bodies w.r.t. symmetric difference metric: Figalli, Maggi, Pratelli, constant improved by Kolesnikov, Milman
- Optimal stability of Brunn-Minkowski for general sets for \( n = 1 \) w.r.t. symmetric difference metric: Freiman
- Optimal stability of Brunn-Minkowski for general sets, \( n = 2 \) w.r.t. symm. diff. metric: van Hintum, Spink, Tiba, 2020
- Stability of Brunn-Minkowski for general sets w.r.t. symmetric difference metric: Figalli, Jerisson \( \sim 2017 \)
Tools for stability of P-L for log-concave functions

- We may assume that \( \int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1 \) and \( \sup f = 1 \)
- We may assume that \( \lambda = \tau = \frac{1}{2} \)
- \( n = 1 \) (Keith Ball, B. \( \sim \) 2010)
  
  \( f, g \) are log-concave probability densities on \( \mathbb{R} \) satisfying

  \[
  \int_{\mathbb{R}} \sup_{z=\frac{1}{2} x + \frac{1}{2} y} \sqrt{f(x)g(y)} \, dz \leq 1 + \varepsilon
  \]

  for \( \varepsilon \in (0, 1) \), then there exists \( w \in \mathbb{R} \) such that

  \[
  \int_{\mathbb{R}} |f(x) - g(x + w)| \, dx \leq c\varepsilon^{\frac{1}{3}} |\log \varepsilon|^{\frac{4}{3}}.
  \]

**Remark**  Error might be of order \( \varepsilon^{\frac{1}{2}} \) (exponent \( \frac{1}{3} \) above can’t be larger than \( \frac{1}{2} \))
Level sets of a log-concave function

**Lemma (Lovasz-Vempala)**

If $\psi$ is a log-concave probability density on $\mathbb{R}^n$ and $t \in (0, 1)$, then

$$V \left( \{ \psi > (1 - t)\|\psi\|_\infty \} \right) \geq \frac{1}{n! + 1} \cdot \frac{t^n}{\|\psi\|_\infty}.$$  

**Lemma (Lovasz-Vempala)**

If $s \in (0, e^{-4(n-1)})$ and $\psi$ is a log-concave probability density on $\mathbb{R}^n$, then

$$V \left( \{ \psi > s\|\psi\|_\infty \} \right) < \frac{2}{n!} \cdot \frac{|\ln s|^n}{\|\psi\|_\infty},$$

$$\int_{\{\psi < s\|\psi\|_\infty\}} \psi < \frac{e^{n-1}}{(n-1)^{n-1}} \cdot s \cdot |\ln s|^{n-1}.$$
Comparing level sets of $f$, $g$, $h$

$$h(z) = \sup_{z = \frac{1}{2} x + \frac{1}{2} y} \sqrt{f(x)g(y)}$$

$$\frac{1}{2} \{ f > t \} + \frac{1}{2} \{ g > s \} \subset \{ h > \sqrt{ts} \} \quad \text{for } t, s > 0$$

$$F(t) = V(\{ f > t \}) \quad G(s) = V(\{ g > s \}) \quad H(r) = V(\{ h > r \})$$

$$\implies H(\sqrt{ts}) \geq \sqrt{F(t) \cdot G(s)}$$

$$\implies \sqrt{\int_0^\infty F \cdot \int_0^\infty G} \leq \int_0^\infty H \leq (1 + \varepsilon) \sqrt{\int_0^\infty F \cdot \int_0^\infty G}$$

$$\implies PL \ n=1 \ & \ BM \ stab \quad \{ f > t \} \text{ and } \{ g > t \} \text{ are } \text{”almost translates”}$$
log-concave function $\mapsto$ convex body

$K = \{ (x, \ln t) \in \mathbb{R}^n : x \in \{ t > \gamma \} \land \gamma \leq t \leq f(x) \}$

$C = \{ (x, \ln t) \in \mathbb{R}^n : x \in \{ g > \gamma \} \land \gamma \leq t \leq g(x) \}$

$L = \{ (x, \ln t) \in \mathbb{R}^n : x \in \{ h > \gamma \} \land \gamma \leq t \leq h(x) \}$

$h(\frac{1}{2}x + \frac{1}{2}y) \geq f(x)^{\frac{1}{2}} g(y)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{1}{2} K + \frac{1}{2} C \subseteq L$

$|V(K)-V(L)|$ = small  \quad $|V(C)-V(L)|$ = small

BM stab $\quad \Rightarrow \quad K, C, L$ are essentially translated
Part 2
Coordinatewise product of unconditional convex bodies

$K, C$ unconditional $((x_1, \ldots, x_n) \in K \implies (\pm x_1, \ldots, \pm x_n) \in K)$

$$K^{1-\lambda} \cdot C^\lambda = \left\{ \left( \pm |x_1|^{1-\lambda}|y_1|^\lambda, \ldots, \pm |x_n|^{1-\lambda}|y_n|^\lambda \right) \right\}$$

$(x_1, \ldots, x_n) \in K \& (y_1, \ldots, y_n) \in C$

Theorem (Bollobas&Leader, Saroglou)
If $K$ and $C$ are unconditional convex bodies and $\lambda \in (0, 1)$, then

$$V(K^{1-\lambda} \cdot C^\lambda) \geq V(K)^{1-\lambda} V(C)^\lambda,$$

with equality $\iff \exists \Phi$ positive definit diagonal matrix s.t. $K = \Phi C$
Stability of Bollobas-Leader

Theorem (B., De)

If \( \tau \in (0, \frac{1}{2}] \), \( \lambda \in [\tau, 1 - \tau] \) and unconditional convex bodies \( K \) and \( C \) in \( \mathbb{R}^n \) satisfy

\[
V(K^{1-\lambda} \cdot C^\lambda) \leq (1 + \varepsilon)V(K)^{1-\lambda}V(C)^\lambda,
\]

then there exists positive definite diagonal matrix \( \Phi \),
\( \det \Phi = V(K)/V(C) \) such that

\[
V(K \Delta (\Phi C)) < c_0 n^n \left( \frac{\varepsilon}{\tau} \right)^{\frac{1}{19}} V(K).
\]
Proof of Bollobas-Leader via Prekopa-Leindler ala Fradelizi, Cordero-Erausquin, Maurey

Theorem (Prekopa, Leindler, Dubuc)

If $\lambda \in (0, 1)$ and $h, f, g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ log-concave satisfy $h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda}g(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \cdot \left(\int_{\mathbb{R}^n} g\right)^{\lambda},$$

equality implying that for $a = \int_{\mathbb{R}^n} f / \int_{\mathbb{R}^n} g$, there exist $w \in \mathbb{R}^n$ such that $f(x) = a^{-\lambda}h(x - \lambda w)$, $g(y) = a^{1-\lambda}h(y + (1 - \lambda)w)$.

To prove Bollobas-Leader, apply Prekopa-Leindler to

$$f(x_1, \ldots, x_n) = 1_K(e^{x_1}, \ldots, e^{x_n})e^{x_1+\ldots+x_n}$$
$$g(x_1, \ldots, x_n) = 1_C(e^{x_1}, \ldots, e^{x_n})e^{x_1+\ldots+x_n}$$
$$h(x_1, \ldots, x_n) = 1_{K^{1-\lambda} \cdot C^\lambda}(e^{x_1}, \ldots, e^{x_n})e^{x_1+\ldots+x_n}$$

For stability Bollobas-Leader, use stability of Prekopa-Leindler for log-concave functions.
Part 3 - Stability of the Log-Minkowski inequality with hyperplane symmetry

support function \( h_K(u) = \max_{x \in K} \langle x, u \rangle \)

\( S_K \) - surface area measure on \( S^{n-1} \) of a convex body \( K \) in \( \mathbb{R}^n \)

\( \partial K \) is \( C^2 \) \( \implies \) \( dS_K = \kappa^{-1} d\mathcal{H}^{n-1} \) (\( \kappa(u) \) = Gaussian curvature)

Minkowski’s first inequality \( \) If \( V(K) = V(C) \), then

\[
\int_{S^{n-1}} h_C \, dS_K \geq \int_{S^{n-1}} h_K \, dS_K.
\]

Equality \( \iff \) \( K \) and \( C \) are translates.

Part 3 ⊂”\( L_p \) Minkowski Inequalities”, initiated by Firey, Lutwak, developed by Colesanti, Milman, Kolesnikov (local theory) Livshyts Chen&Huang&Li and Putterman (”continuity method”)
Logarithmic Minkowski conjecture

\[ dV_K = \frac{1}{n} h_K dS_K \] - cone volume measure on \( S^{n-1} \) if \( o \in K \)

(Firey, 1974, Gromov, Milman, 1986) = \( L_0 \) surface area measure

Conjecture (B, Lutwak, Yang, Zhang)

If \( K \) and \( C \) are convex bodies whose centroid is the origin and \( V(K) = V(C) \), then

\[
\int_{S^{n-1}} \log h_C \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K.
\] (1)

Assuming \( K \) is smooth, equality holds \( \iff K = C \).

Known results

- \( K \) is close to some ellipsoid (Colesanti&Livshyts&Marsiglietti, Kolesnikov&Milman, Chen&Huang&Li&Liu)
- \( K, C \) have complex symmetry (Rotem)
- \( K, C \) - hyperplane symmetry (Saroglou, B&Kalantzopoulos)
Coinciding cone volumes

\[ V_k = V_C \]

\[ \frac{1}{a} \]

\[ C \]
Convex bodies with "hyperplane symmetries"

Linear reflection: \( A \in \text{GL}(n), A \neq \text{Id}, A^2 = A \) and there exists a linear hyperplane \( H \) s.t. \( Ax = x \) for \( x \in H \).

A convex body \( K \) has "hyperplane symmetries" \( \iff \) \( K \) is invariant under some linear reflections \( A_1, \ldots, A_n \) through hyperplanes \( H_1, \ldots, H_n \) with \( H_1 \cap \ldots \cap H_n = \{o\} \) \( \iff \) \( K \) invariant under a Coxeter group \( G \subset \text{GL}(n) \) of rank \( n \)

- Idea comes from
  - Barthe & Fradelizi’s work on Mahler’s conjecture
  - Barthe & Cordero-Erausquin’s work on the Slicing conjecture
- \( G \) has a simplicial cone \( C \) as fundamental domain, and reflections through the walls of \( C \) generate \( G \)
- \( C \) is mapped into a "coordinate corner" by a linear transform, and results about unconditional bodies are used.
Convex body $K$ with symmetries of a regular simplex
Stability of Log-Minkowski with hyperplane symmetries

Theorem (B., De)

If the convex bodies $K$ and $C$ in $\mathbb{R}^n$ are invariant under the Coxeter group $G \subset \text{GL}(n)$ generated by $n$ independent linear reflections, and

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \frac{dV_K}{V(K)} \leq \frac{1}{n} \cdot \log \frac{V(C)}{V(K)} + \varepsilon$$

for $\varepsilon > 0$, then for some $m \geq 1$, there exist compact convex sets $K_1, C_1, \ldots, K_m, C_m$ of dimension at least one and invariant under $G$ where $K_i$ and $C_i$ are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^{m} \dim K_i = n$ such that

$$K_1 + \ldots + K_m \subset K \subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right) (K_1 + \ldots + K_m)$$

$$C_1 + \ldots + C_m \subset C \subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right) (C_1 + \ldots + C_m)$$

where $c > 1$ is an absolute constant.