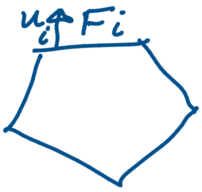
A non-existence result for the L_p-Minkowski problem

K convex body

$$S_K(\omega) = \mathcal{H}^{n-1}(\{x \in \partial K : \exists u \in \omega, \langle u, x\rangle = h_K(u)\})$$

$\omega \subseteq S^{n-1}$.



$$S_K(\cdot) = \sum_{i=1}^{m} |F_i| \delta_{u_i}(\cdot).$$

Minkowski problem: Let μ is Borel measure on S^{n-1}. Find necessary and sufficient conditions so that $\mu = S_K$, for some K.

Answer: Iff μ is not concentrated in a proper subspace of $\mathbb{R}^n$ and

$$\int_{S^{n-1}} \vec{x} \, d\mu = 0.$$

L_p-surface area measure of K ($K \ni 0$):

$$dS_{K,p}(\cdot) = h_K^{1-p} \, dS_K(\cdot).$$

$$|K+tL|'_{t=0} = \frac{1}{n}\int_{S^{n-1}} h_L \, dS_K$$

$$|K+_p tL|'_{t=0} = \frac{1}{n}\int_{S^{n-1}} h_L^p \, dS_{K,p}.$$

<u>L_p-Minkowski problem</u>: Let μ as above

$$S_{K,p} = \mu$$

↳ when is it solvable

$(K \ni o)$

$\boxed{\text{Not } L^p\text{-Brunn-Minkówski}}$

Initiated by Lutwak ('93)

Huang-Lutwak-Yang-Zhang (2005)

Chou-Wang (2006)

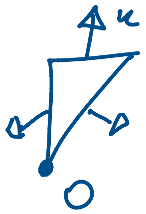
Böröczky-Lutwak-Yang-Zhang (2012, $p=0$ even case)

Chen-Li-Zhou (2019, general $p=0$, sufficient condition).

<u>Case $0<p<1$:</u> Chen-Li-Zhu (2017)

If μ is not concentrated in a proper subspace, then $\exists K$, s.t. $S_{K,p} = \mu$.

__Example:__



$$\underset{\overset{\|}{h_K^{1-p}\, dS_K}}{S_{K,p}(\cdot)} = c \cdot \delta_u(\cdot)$$

__Conjecture__ (Bianchi - Böröczky - Colesanti - Yang, 2019): If $0 < p < 1$, then $S_{K,p} = \mu$ is solvable iff μ is not concentrated in a pair of antipodal points.

- Yes, if $n = 2$. (Böröczky - Trinh (2020))
- __Thm__: No, if $n \geq 3$.

__Example__: μ is concentrated in proper subspace H and $\mu \llcorner S^{n-1} \cap H$ = spherical Lebesgue measure.

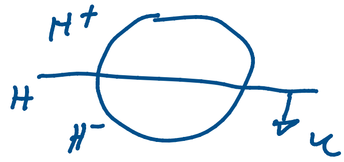
__Question__: μ is concentrated in H and $\mu \llcorner S^{n-1} \cap H = f\, dx$, $f > c > 0$. Is there such f, so that

$$\mu = S_{K,p} \text{ is solvable?}$$

__LL__: Let $\alpha < 0$, $o \in \partial K$, $h_K(u) = 0$, for some $u \in S^{n-1}$. Then,

$$\int_{S^{n-1}} \langle u, v\rangle\, h_K(v)^q \, dv > 0 .$$

<u>Proof</u> : Polar coordinates



$$\int_{B_2^n \cap H^\pm} \langle x, u\rangle\, h_K(x)^q\, e^{-\frac{1}{|x|}}\, dx$$

$$= \int_{S^{n-1} \cap H^\pm} \langle u, v\rangle\, h_K(v)\, dv \cdot \underbrace{\int_0^1 r^{n+q} e^{-1/r}\, dr}_{\underset{c}{\parallel}}$$

$$= c \int_{S^{n-1} \cap H^\pm} \langle u, v\rangle\, h_K(v)\, dv.$$

- It suffices

$$\int_{B_2^n \cap H^+} \langle x, u\rangle\, h_K(x)^q\, e^{-\frac{1}{|x|}}\, dx >$$

$$\int_{B_2^n \cap H^-} \langle x, u\rangle\, h_K(x)^q\, e^{-\frac{1}{|x|}}\, dx \qquad (*)$$

- But $\underbrace{\boxed{h_K(z+tu) \le h_K(z-tu)}}_{(**)} + 2t\, h_K''(u)$,

$\forall\, t > 0,\ z \in u^\perp$

$$\Rightarrow \int_{[B_2^n \cap (u^\perp + tu)] - tu} h_K(z+tu)^q\, e^{-\frac{1}{|z+tu|}}\, dz$$

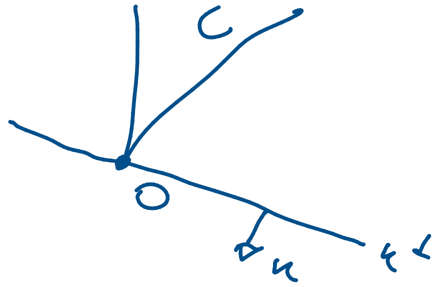
$< \int \ldots \ldots \quad h_u(z-tu)^q e^{-\frac{1}{|z-tu|}} dz$

Fubini $\Longrightarrow$ $\circledast$ holds with "$\geq$" instead of "$>$".

Equality cases in $\circledast\circledast$ $\Longrightarrow$ $\circledast$ holds. $\blacksquare$

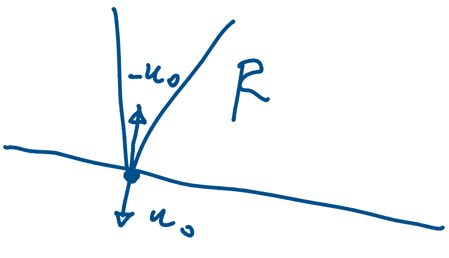
- $C \subset \mathbb{R}^n$ is a convex if $ax+by \in C$, $\forall\, a,b \geq 0$, $\forall\, x,y \in C$.

$u \in S^{n-1}$ supports C (always at 0)



if $\langle u, x\rangle \leq 0$, $\forall x \in C$

__L2:__ Let R be closed convex in $\mathbb{R}^n$, supported by at least one unit vector. Then, $\exists\, u_0 \in S^{n-1} \cap (-R)$, s.t. u_0 supports R.



L3: If $\underset{\substack{\cap \\ S^{n-1}}}{H^-(v}, \underset{\substack{\cap \\ \mathbb{R}}}{t)} := \{x : \langle x, v\rangle \le t\}$,

then $K = \bigcap_{v \in \operatorname{supp}(K,\cdot)} H^-(v, h_K(v))$.

L4: Let $p < 1$, $H \le \mathbb{R}^n$, $1 \le \dim H \le n-1$,
K be s.t. $\operatorname{supp}(S_p(K,\cdot)) \subseteq H$.
Set $S := (\operatorname{supp}(S(K,\cdot))) \setminus H$
$T := \{h_K = 0\}$.

Then, $\exists u_0 \in S^{n-1} \cap H$ and $y \in H$, s.t.

(i) $h_{K|H}(u_0) = 0$
$\hookrightarrow$ orthogonal projection
$\Updownarrow$
$h_K(u_0) = 0$

(ii) $\langle u_0, v\rangle \geq 0$, $\forall v \in T \cap H$

(iii) $\langle u_0 + y, v\rangle \geq 0$, $\forall v \in S$

Proof:
- $S = \emptyset$
- $S \subseteq T$
- $K = C \cap P$

where $C = \bigcap_{v \in T} H^-(v, 0)$

$P = \bigcap_{v \in S^{n-1} \cap H} H^-(v, h_K(v))$

• $\exists u \in S^{n-1} \cap H$, s.t. u supports C/H

$\xRightarrow{L2}$ $\exists u_0 \in S^{n-1} \cap H$, s.t. u_0 supports C/H and $-u_0 \in C/H$.

$\Rightarrow h_{K/H}(u_0) = 0$.

• $\exists y \in H^\perp$, s.t. $w := -u_0 - y \in C$

$\Rightarrow \langle w, v \rangle \le 0$, $\forall v \in T$

$\Rightarrow \langle u_0 + y, v \rangle \geq 0$, $\forall v \in S \subseteq T$.

Part (ii) trivially follows from (iii).

<u>Proof of Thm:</u> Let K, s.t.

$$S_p(K,\cdot) = \mu$$

Assumption $\Rightarrow \int_{(S^{n-1}\cap H)\setminus T} \varphi \, dS_K(\cdot)$

$$= \int_{(S^{n-1}\cap H)\setminus T} h_K^{-(1-p)} \, du$$

$$= \int_{S^{n-1}\cap H} \varphi \, h_K^{-(1-p)} \, du, \quad \forall \varphi : (S^{n-1}\cap H)\setminus T \to \mathbb{R}.$$

- $\mathcal{H}^{k-1}(T\cap H) = 0$

- $\int_{S^{n-1}\cap H} \langle u_0, v\rangle \, dS(K,v)$

$$= \int_{S^{n-1}\cap H} \langle u_0, v\rangle \, h_{K|H}^{-(1-p)} \, dv$$

$\overset{h_{K|H}(u_0)=0}{>} 0.$

$\angle \perp$

- $\int_{S^{n-1}} \langle u_0 + y, v\rangle \, dS(K,v)$

$$= \int_{\substack{S \\ \shortparallel \\ \operatorname{supp}(S(K,\cdot))\setminus H}} \overset{1,0}{\cdots} + \int_{S^{n-1}\cap H} \overset{1\,0}{\cdots}$$

$\operatorname{supp}^{\pi}(S(K,\cdot)) \setminus H$ $\quad S^{n-1} \cap H$

> 0

Contradiction because of Minkowski's Thm.

Thank you !!!