Covariance Representations on the Sphere

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Gaussian Covariance Identity

- Let μ denote the standard Gaussian measures on $\mathbb{R}^n$ with density $\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}}, x \in \mathbb{R}^n$

Theorem 1

For a unique probability measure λ on $\mathbb{R}^n \times \mathbb{R}^n$, with marginals μ and u, v arbitrary smooth functions on R^n with μ-square integrable gradients

$$Cov_\mu(u, v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} < \nabla u(x), \nabla v(y) > d\lambda(x, y)$$

- Uniqueness: Consider $u(x) = e^{i<t,x>}, v(y) = e^{i<s,y>} (t, s \in \mathbb{R}^n)$
- Marginals: Consider $v(y) = \sum_{i\geq 1}^{n} y_i$

Characterisation of Gaussian measures

Theorem 2

Given a probability measure μ on $\mathbb{R}^n$, suppose that

$$Cov_\mu(u, v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} < \nabla u(x), \nabla v(y) > d\lambda(x, y)$$

for some finite measure λ on $\mathbb{R}^n \times \mathbb{R}^n$ in the class of all bounded smooth u, v on $\mathbb{R}^n$ with bounded partial derivatives. Then μ is Gaussian with covariance matrix $\sigma^2 I_n$.

- Proof: An application of the Darmois-Skitovitch theorem
- Question: May such a representation exist in different spaces?

One such derivation

- Ledoux 1995

$$Cov_\mu(u,v) = \int_0^\infty \mathbb{E}_\mu < \nabla u, \nabla Q_t v > dt$$

where $Q_t v(x) = \int_{\mathbb{R}^n} v(e^{-t}x + \sqrt{1-e^{-2t}}y)d\mu(y)$ is the Ornstein-Uhlenbeck semigroup associated to $L_{O.U.} = \Delta - x \cdot \nabla$

- Note that for v such that $\mathbb{E}_\mu v = 0$

$$v = Q_0 v - Q_\infty v = -\int_0^\infty \frac{d}{dt} Q_t v dt = -\int_0^\infty L_{O.U.} Q_t v dt$$

- Thus

$$Cov_\mu(u,v) = \int_{\mathbb{R}^n} uv d\mu = \int_0^\infty \mathbb{E}_\mu < \nabla u, \nabla Q_t v > dt$$

One such derivation

- Using the integral representation of Q_t, one obtains

$$Cov_\mu(u,v) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} < \nabla u(x), \nabla v(y) > d\lambda(x,y)$$

  with λ admitting a density
  $p(x,y) = \frac{1}{(2\pi)^n}\int_0^1 s^{-n} exp(-\frac{|x|^2+|y|^2-2t<x,y>}{2s^2})dt$ where $s = \sqrt{1-t^2}$
- Interpolation Approach: Houdré, Pérez-Abreu 1995, H-PA-Surgailis 1998, Bobkov-H-Götze 2001

$$Cov_\mu(f,g) = \int_0^1 \mathbb{E} < \nabla f(X), \nabla g(tX+sZ) > dt$$

$$X \perp Z \sim \mu$$

Concentration in Gauss Space

- Sobolev-type inequality:

$$Cov_\mu(u, v) \leq ||\nabla u||_{L^p(\mu)} ||\nabla v||_{L^q(\mu)}$$

- With $q = 1, v = e^{tu}, ||u||_{Lip} \leq 1$

$$\mathbb{E}_\mu e^{tu} \leq e^{t\mathbb{E}_\mu u + \frac{t^2}{2}}$$

$$\Rightarrow \mu\{|u - \mathbb{E}_\mu u| \geq h\} \leq 2e^{-\frac{h^2}{2}}$$

- Using the entropy functional

$$\mathbb{E}_\mu e^g \leq \mathbb{E}_\mu e^{|\nabla g|^2 + \mathbb{E}_\mu g}$$

- Corollary: For $h > 0, ||u||_{Lip} \leq 1$

$$\mu\{|u - \mathbb{E}_\mu u| \geq h\} \leq \frac{e^{-\frac{h^2}{2}}}{h} \mathbb{E}_\mu |u - \mathbb{E}_\mu u|$$

Proof of Corollary

Let v=min$\{$(u-h)$^+, \epsilon\}, h, \epsilon > 0$ for $\mathbb{E}_\mu u = 0$. Let p denote the density of u under μ.

The representation gives

$$\mathbb{E}_\mu u T(u) \leq \mathbb{E}_\mu T'(u)$$

$$\int_h^{h+\epsilon} x(x-h)p(x)dx + \epsilon \int_{h+\epsilon}^\infty xp(x)dx \leq \int_h^{h+\epsilon} p(x)dx$$

Dividing by and letting $\epsilon \to 0$

$$V(h) := \int_h^\infty xp(x)dx \leq p(h) = -\frac{V'(h)}{h}$$

Monotonicity gives

$$V(h)e^{\frac{h^2}{2}} \leq V(0) = \mathbb{E}_\mu u^+$$

Using $V(h) \geq h\mu\{u \geq h\}$,

$$\mu\{u \geq h\} \leq \frac{e^{-\frac{h^2}{2}}}{h}\mathbb{E}_\mu u^+$$

Spherical Derivatives

- Consider $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}, n \geq 2$
- Let σ_{n-1} denote the uniform measure on the sphere
- $w = \nabla_s f(\theta)$ at $\theta \in \mathbb{S}^{n-1}$ denotes the shortest vector such that for $\theta' \in \mathbb{S}^{n-1} \to \theta$

$$f(\theta') = f(\theta) + <w, \theta' - \theta> + o(|\theta' - \theta|)$$

- If f is smooth in a neighbourhood of the sphere

$$\nabla_s f(\theta) = P_{\theta^\perp} \nabla f(\theta) = \nabla f(\theta) - <\theta, \nabla f(\theta)> \theta$$

Spherical Derivatives

- Define $B = f_s''(\theta)$ is the symmetric nxn matrix with the smallest Hilbert-Schmidt norm such that for $\theta' \in \mathbb{S}^{n-1} \to \theta$

$$f(\theta') = f(\theta) + < \nabla_s f(\theta), \theta' - \theta > + \frac{1}{2} < B(\theta' - \theta), \theta' - \theta > + o(|\theta' - \theta|^2)$$

- $\Delta_s f(\theta) = \mathrm{Tr} f_s''(\theta)$
- For smooth $f, g : \mathbb{S}^{n-1} \to \mathbb{R}$, associate functions $u, v : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$

$$u(x) = f(r^{-1}x) = f(\theta), v(x) = g(r^{-1}x) = g(\theta), r = |x|$$

- One has

$$\nabla u(x) = \frac{1}{r} \nabla_s f(\theta), \nabla v(x) = \frac{1}{r} \nabla_s g(\theta)$$

Construction of the spherical representation

- Note that $Cov_{\sigma_{n-1}}(f,g) = Cov_{\mu}(u,v)$
- By applying the Gaussian covariance identity to u and v and integrating in polar coordinates we write $Cov_{\sigma_{n-1}}(f,g)$ as

$$\int_{S^{n-1}} \int_{S^{n-1}} < \nabla_s f(\theta), \nabla_s g(\theta') > \Psi_{n-1}(< \theta, \theta' >) d\sigma_{n-1}(\theta) d\sigma_{n-1}(\theta')$$

$$\Psi_{n-1}(\alpha) = \frac{1}{2^{n-2}\Gamma(\frac{n}{2})^2} \int_0^1 s^{n-2} \left[\int_0^\infty \int_0^\infty \exp\left[-\frac{r^2+r'^2-2rr't\alpha}{2} \right] (rr')^{n-2}\, dr\, dr' \right] dt.$$

Study of mixing measure

- By virtue of the rotational invariance, define

$$c_{n-1} = \int_{\mathbb{S}^{n-1}} \Psi_{n-1}(< \theta, \theta' >) d\sigma_{n-1}(\theta')$$

- By Cauchy-Bunyakovsky, c_{n-1} must be larger than $\frac{1}{n-1}$ which is optimal in the Poincaré inequality
- By polar integration,

$$c_{n-1} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{p(x,y)}{|x||y|} dxdy \leq \mathbb{E}_\mu \frac{1}{|X|^2} = \frac{1}{n-2}$$

Spherical covariance representation

Theorem 3

On $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, n \geq 3$ *there exists a probability measure* ν *with marginals* σ_{n-1} *such that*

$$Cov_{\sigma_{n-1}}(f, g) = c_{n-1} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} < \nabla_s f(\theta), \nabla_s g(\theta') > d\nu(\theta, \theta')$$

where $\frac{1}{n-1} < c_{n-1} < \frac{1}{n-2}$

- Uniquenes is not ensured. Consider $S \subset \mathbb{S}^{n-1}$ a circle and ν' a measure supported on the set $A = \{(x, y) \in S \times S :< x, y >= 0\}$. $c_{n-1}\nu + \nu'$ will satisfy this representation.

Spherical Concentration

By previous arguments, we have the following spherical analogues

$$Cov_{\sigma_{n-1}}(f,g) \leq c_{n-1}||\nabla_s f||_{L^p(\sigma_{n-1})}||\nabla_s g||_{L^q(\sigma_{n-1})}$$

$$\mathbb{E}_{\sigma_{n-1}} e^f \leq \mathbb{E}_{\sigma_{n-1}} e^{c_{n-1}|\nabla f|^2 + \mathbb{E}_{\sigma_{n-1}} f}$$

- Classic Deviation Inequality for $||f||_{Lip} \leq 1, h > 0$

$$\sigma_{n-1}\{|f - \mathbb{E}_{\sigma_{n-1}} f| \geq h\} \leq 2e^{-(n-1)\frac{h^2}{2}}$$

- Corollary: For $h > 0, ||f||_{Lip} \leq 1$

$$\mu\{|f - \mathbb{E}_{\sigma_{n-1}} f| \geq h\} \leq \frac{e^{-\frac{h^2}{2c_{n-1}}}}{h}\mathbb{E}_{\sigma_{n-1}}|f - \mathbb{E}_{\sigma_{n-1}} f|$$

Heat Semigroup

- Consider the heat semigroup $P_t = e^{t\Delta_s}$
- The semigroup is frequently defined in the two following ways

  1) Consider $f = \sum_{d \geq 0} f_d$ its decomposition into spherical harmonics, then

  $$P_t f = \sum_{d \geq 0} e^{-d(d+n-2)t} f_d$$

  2) $P_t f(\theta) = \mathbb{E}[f(B_{2t})|B_0 = \theta]$ where B_t is brownian motion on $\mathbb{S}^{n-1}$
- There exists a non-negative and non-decreasing $K_t \in \mathcal{C}^\infty$ on [-1,1] such that $P_t f(\theta) = \int_{\mathbb{S}^{n-1}} K_t(< \theta, \theta' >) f(\theta') d\sigma_{n-1}(\theta')$
- $P_t f(\theta) \to \mathbb{E}_{\sigma_{n-1}} f, t \to \infty$
- $P_t f(\theta) \to f, t \to 0$
- $\frac{d}{dt} P_t f = \Delta_s P_t f = P_t \Delta_s f$

Semigroup representation

- By integration by parts on the sphere, one has the following using the argument above for f,g smooth and $n \geq 3$

Theorem 4

$$Cov_{\sigma_{n-1}}(f,g) = \int_0^\infty \int_{\mathbb{S}^{n-1}} < \nabla_s f(\theta), \nabla_s P_t g(\theta) > d\sigma_{n-1}(\theta) dt$$

- Using properties of spherical harmonics and hermite polynomials, Theorem 4 furnishes Theorem 1!

Extension to other semigroups

- Consider U_t an ergodic Markovian semigroup associated to a probability measure μ over $\mathcal{E}$.
- Ergodicity is ensured if μ is finite and $Lf = 0 \to f$ constant.
- By the argument above

$$Cov_\mu(f, g) = \int_0^\infty \int_{\mathcal{E}} \Gamma(f, U_t g) d\mu dt$$

- Example: $\frac{d\mu}{dx} = \frac{1}{Z} e^{-V(x)}$, $Hess V(x) \geq \rho > 0$

$$Cov_\mu(f, g) = \int_0^\infty \int_{\mathbb{R}^n} < \nabla f, \nabla U_t g > d\mu dt$$

Log-Concave case

- Recall that the example above satisfies the $CD(\rho, \infty)$ condition.

$$\Rightarrow |\nabla U_t g| \leq e^{-\rho t} U_t |\nabla g|$$

- By similar arguments as above in conjunction with reflexivity and Jensen's inequality

$$Cov_\mu(f,g) \leq \frac{1}{\rho} ||\nabla f||_{L^p(\mu)} ||\nabla g||_{L^q(\mu)}$$

$$h > 0, ||f||_{Lip} \leq 1, \mu\{|f - \mathbb{E}_\mu f| \geq h\} \leq \frac{e^{-\frac{h^2 \rho}{2}}}{h} \mathbb{E}_\mu |f - \mathbb{E}_\mu f|$$

Second Order Semigroup Argument

- On the sphere, integrate by parts with respect to time

$$Cov_{\sigma_{n-1}}(f,g) = \int_0^\infty \int_{\mathbb{S}^{n-1}} t\Delta_s P_t f \Delta_s g d\sigma_{n-1} dt = \int_0^\infty \int_{\mathbb{S}^{n-1}} tP_t f \Delta_s^2 g d\sigma_{n-1} dt$$

- Apply latter representation to spherical harmonics and use formula

$$\int_{\mathbb{S}^{n-1}} ||f_s''||_{HS}^2 d\sigma_{n-1} = \int_{\mathbb{S}^{n-1}} f\Delta_s^2 f + (n-2)\Delta_s f d\sigma_{n-1}$$

Recovers

$$\int_{\mathbb{S}^{n-1}} f^2 d\sigma_{n-1} \leq \frac{1}{n-1} \int_{\mathbb{S}^{n-1}} ||f_s''||_{HS}^2 d\sigma_{n-1}$$

for $\mathbb{E}_{\sigma_{n-1}} f = 0$

Second Order Argument

Let us return to the Gaussian case

- Applying the Gaussian covariance identity to the derivatives of u and v

$$\to Cov_{\mu}(u,v) = < \mathbb{E}_{\mu}\nabla u, \mathbb{E}_{\mu}\nabla v > + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} < u''(x), v(y)'' > dk_n(x,y)$$

where $2k_n$ is a probability measure with marginals μ

- Similar result attainable by approach of Ledoux?
- As above, let us associate u,v to functions f,g on the sphere
- One has

$$u''(x) = r^{-2}Df(\theta) = r^{-2}(f_s''(\theta) - 2\nabla_s f(\theta) \otimes \theta)$$

$$(\alpha \otimes \beta)_{ij} = \frac{1}{2}(\alpha_i\beta_j + \beta_i\alpha_j), 1 \leq i,j \leq n$$

Second Order Argument

- Assume $\mathbb{E}_\mu \nabla u = \mathbb{E}_\mu \nabla v = 0$
  Equivalently, f, g are orthogonal to linear functions

Theorem 5

On $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, n \geq 5$ there exists a probability measure ν with marginals σ_{n-1} such that

$$Cov_{\sigma_{n-1}}(f,g) = d_{n-1} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} < Df(\theta), Dg(\theta') > d\nu(\theta, \theta')$$

for f,g orthogonal to linear functions where $\frac{1}{n(n+2)} < d_{n-1} < \frac{1}{(n-2)(n-4)}$

- Furnishes the following sharpening for f orthogonal to affine functions

$$\mathbb{E}_{\sigma_{n-1}} e^f \leq \mathbb{E}_{\sigma_{n-1}} exp\left(\frac{1}{(n-2)(n-4)}(2||f_s''||_{HS}^2 + 8|\nabla_s f|^2)\right)$$