HOME WORK I, DIFFERENTIAL GEOMETRY: ADVANCED BRIEF REVIEW OF THE DIFFERENTIAL GEOMETRY OF SURFACES IN $\mathbb{R}^n$.

Due January 28. The Home Work must be uploaded on Canvas as a pdf. To complete the home work, review the topics of differential geometry of curves and surfaces, using e.g. DoCarmo's Differential Geometry of Curves and Surfaces, chapters 1, 2, 3, 4.1-4.5, as well as the lecture notes of the first three lectures. Please contact me if you have any questions!

1. a) Find a suitable parametrization of the open upper half sphere

$$X : \{y \in \mathbb{R}^{n-1} : |y| < 1\} \to \mathbb{S}^{n-1} \cap \{x : x_n > 0\}.$$

Next, **write out the general definitions** and compute in this particular case the following:
b) Give the expression for the Gauss map at a generic point $p \in \mathbb{S}^{n-1} \cap \{x : x_n > 0\}$;
c) Write out the shape operator S_p;
d) Compute the Gauss curvature at p;
e) Is the sphere locally convex at p? An answer and a short explanation suffices;
f) Compute the first fundamental form at p;
g) Find the area of $\mathbb{S}^{n-1} \cap \{x_n > \frac{1}{\sqrt{n}}\}$ using the area element expression in terms of the Gauss curvature, and choosing the correct domain of integration. It suffices to express this area in terms of the $(n-1)$–dimensional volume of $B_2^{n-1} = \{y \in \mathbb{R}^{n-1} : |y| \leq 1\}$.
h) Compute the second fundamental form at p;
i) Compute the normal curvature at p in a generic direction v;
j) Find the principal curvatures (the eigenvalues of the shape operator).
k) For the parametrization you chose, write out the moving frame at p;
l) Compute the Christofel symbols Γ_{ij}^k and l_i^j.

2. Compute the first fundamental form of the following surfaces in $\mathbb{R}^3$:
a) Ellipsoid $X(x, y) = (a \sin x \cos y, b \sin x \sin y, c \cos x)$.
b) Saddle $X(x, y) = (x, y, xy)$.

3. Recall that the second fundamental form of a surface M in $\mathbb{R}^n$ parametrized by $X : U \subset \mathbb{R}^{n-1} \to V \subset M$ at a point $p \in V$ was defined as $l_{ij} = \langle D_{ij}X, N \rangle$, where $N = n_p \circ X^{-1}$ is the unit normal (Gauss map) at p, defined on U. Suppose $0 \in U$ and $X(0) = p$. Prove that

$$l_{ij}(0) = \langle S_p(D_i X(0)), D_j X(0) \rangle,$$

where S_p is the shape operator at p, that is for $x \in T_p M$, $S_p(x) = -dn_p(x) = -J_{n_p} x$.

4. Compute the Gauss map and the second fundamental form at $p = (0,0)$ for the saddle surface $X(x,y) = (x,y,xy)$ in $\mathbb{R}^3$. Use either the definition or the formula from the previous exercise, up to you.

5. Find the principal curvatures at $p = (0,0)$ for the saddle surface $X(x,y) = (x,y,xy)$ in $\mathbb{R}^3$.