HOME WORK II, DIFFERENTIAL GEOMETRY: GLOBAL INEQUALITIES AND CONVEXITY.

Due February 6. The Home Work must be uploaded on Canvas as a pdf. In addition to the class notes, you may find helpful the books by Schneider "the Brunn-Minkowski theory" and Bonnesen, Fenchel "Theory of Convex bodies". Assignments marked with asterisk are optional, and furthermore, may be treated as optional reading assignments (for the source, see the aforementioned books). In other words, only questions 1, 5 and 6 are mandatory. Please contact me if you have any questions!

1. Deduce Minkowski's first inequality from the Brunn-Minkowski inequality: show that for any pair K and L of convex bodies in $\mathbb{R}^n$,

$$V_1(K, L) \geq |K|^{\frac{n-1}{n}} |L|^{\frac{1}{n}}. \tag{0.1}$$

Hint: recall the definition of the mixed volume $V_1(K, L)$ and review the proof of the isoperimetric inequality which we did in class.

2^*. Conclude the proof of Minkowski's uniqueness theorem which we discussed. That is, show that the equality in the Brunn-Minkowski inequality

$$|\lambda K + (1-\lambda)L|^{\frac{1}{n}} \geq \lambda |K|^{\frac{1}{n}} + (1-\lambda)|L|^{\frac{1}{n}}$$

is achieved if and only if K and L are convex, and furthermore $K = tL$, for some $t > 0$, in the case when K and L have non-empty interior. Deduce that the equality in (0.1) for compact convex sets K and L with non-empty interior may only occur only when $K = tL$.

3^*. Show that the Gauss curvature $\kappa(x)$ of a smooth convex set K at a point $x \in \partial K$, with $\nu_K(x) = u \in \mathbb{S}^{n-1}$, is given by $\det Q(h(u))$, with the matrix Q defined in class (in terms of a moving frame on $\mathbb{S}^{n-1}$).

4.* Show that the Brunn-Minkowski inequality implies the "mean width inequality". That is, of all the convex sets K with *the mean width*

$$W(K) := \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} h_K(u)du = 1,$$

the unit ball has the largest volume. Hint: note first that if $W(K) = W(L) = 1$, then for every $\lambda > 0$, we have $W(\lambda K + (1-\lambda)L) = 1$.

5. Find the support function of
a) $B^n_\infty = \{x \in \mathbb{R}^n : \max_{i=1,...,n} |x_i| \leq 1\}$;
b) $B^n_1 = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \leq 1\}$.

6. Given a unit vector $v \in \mathbb{S}^{n-1}$, find the $(n-1)$-dimensional Lebesgue measure of the projection of B^n_∞ (defined above) onto the hyperplane $v^\perp = \{x \in \mathbb{R}^n : \langle x, v\rangle = 0\}$.

Hint: use Cauchy's projection formula which we discussed in class. In the particular case of a polytope K with areas of faces equal to $F_1, ..., F_N$ and normals to faces $u_1, ..., u_N \in \mathbb{S}^{n-1}$, it gives

$$|K|v^\perp|_{n-1} = \frac{1}{2}\sum_{i=1}^{N} |\langle v, u_i\rangle| F_i.$$