HOME WORK III, DIFFERENTIAL GEOMETRY: DIFFERENTIABLE MANIFOLDS.

Due February 13. The Home Work must be uploaded on Canvas as a pdf. In addition to the class notes, you may find helpful the book by DoCarmo "Riemanian geometry". Problems marked with an asterisk are optional. Please contact me if you have any questions!

1. Let M be an $m-$dimensional differential manifold, and N be another $n-$dimensional manifold. Define the $(m+n)-$dimensional product manifold

$$M \times N = \{(x_1, ..., x_m, y_1, ..., y_n) : x \in M, \ y \in N\},$$

and consider a family of maps, for $(p, q) \in U_\alpha \times V_\beta$, given by

$$z_{\alpha,\beta}(p, q) = (x_\alpha(p), x_\beta(q)),$$

where (x_α, U_α) and (y_β, V_β) are corresponding differentiable structures on M and N.
a) Prove that $(z_{\alpha,\beta}, U_\alpha \times V_\beta)$ form a differentiable structure.
b) Define projection map $\pi : M \times N \to M$ by $\pi(x_1, ..., x_m, y_1, ..., y_n) = (x_1, ..., x_m)$. Show that this map is differentiable.

2. Show that if differentiable manifolds are diffeomorphic, and one of them is orientable, than the other one is also orientable.

3.* Show that the projective plane $P^2(\mathbb{R})$ is non-orientable. Hint: note that $P^2(\mathbb{R})$ contains an open subset diffeomorphic to a Mobius band.

4. Show (using the inverse function theorem) that for every immersion φ of an $n-$dimensional

differentiable manifold M_1 into an $m-$dimensional differentiable manifold M_2, where $n \leq m$, for every point $p \in M_1$ there exists an open neighborhood $V \subset M_1$, such that $\varphi|_V$ is an embedding.