# Analysis 2 Home work 4

March 4, 2019

Due March 27. Questions marked with an asterisk are optional, but highly recommended! Good luck!

**1.** For any $\epsilon \in [0, 1]$, find an upper estimate on the integer $n$ such that there exists an approximation of $f(x) = |x|$ on $[-1, 1]$ by a polynomial $P(x)$ of degree $n$ such that

$$\sup_{x\in[-1,1]} |P(x) - |x|| \leq \epsilon.$$

In other words, you need to find an upper estimate on the integer-valued function $n(\epsilon)$. (note, you are not requested to find "the very best possible" estimate — just any one will be good!)

**2.** a) For any $\epsilon \in [0, 1]$, find an upper estimate on the integer $n$ such that there exists an approximation of $f(x) = \log x$ on $[1, 2]$ by a polynomial $P(x)$ of degree $n$ such that

$$\sup_{x\in[1,2]} |P(x) - \log x| \leq \epsilon.$$

In other words, you need to find an upper estimate on the integer-valued function $n(\epsilon)$.

b) Consider two possible ways of achieving the previous task: via evaluating the proof of the Wejerstrass theorem, and via the Taylor's theorem. Compare the results.

c)* What about the lower estimate on $n(\epsilon)$ — any thoughts?

**3.** a) Show that

$$\sum_{k=1}^{\infty} \frac{1}{(n+k)!} < \frac{1}{n!}.$$

b) Conclude that $e$ is an irrational number, using Taylor's theorem.

4.* Does there exist an infinitely differentiable real-valued function which is not real-analytic?

**5.** Recall that a function is called convex on $E$ (which is a subset of a linear space) if for every $x, y \in E$ and every $\lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. Prove that if a function is twice differntiable, then it is convex if and only if $f''(x) \leq 0$.

**6.** a) For any real number $\theta$ and every integer $n$, show the de Moivre formula

$$\cos(n\theta) = Re\,(\cos\theta + i\sin\theta)^n\,,$$

$$\sin(n\theta) = Im\,(\cos\theta + i\sin\theta)^n\,.$$

b) Compute $\cos(7\theta)$ in terms of the trigonometric functions of $\theta$.

**7.** a) Consider the function $f(x) = (1-2x)^2$ on $[0,1]$ and extend it periodically.
B) Show that the series

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\cos(2\pi n x)$$

converges uniformly to $f$.
C) Find exactly the value $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

**8.** Suppose the function $f$ is differentiable and $\frac{1}{2}$-periodic (i.e., more than just 1-periodic). Show that the Poincare inequality (which we proved in class) holds with a better constant:

$$\int_0^1 f^2(x)dx - \left(\int_0^1 f(x)dx\right)^2 \leq \pi^2 \int (f'(x))^2 dx.$$

**9.** Decompose $f(x) = 1_{[0,\frac{1}{2}]}(x)$ into Fourier series.

**10.** Find the Fourier transform (on the line) of $f(x) = \frac{\sin x}{x}$.