HOME WORK 6, ANALYSIS I

Due November 5.

1. Finish the proof of the Cauchy criterion: recall that a_n is a decreasing sequence of non-negative real numbers, and

$$S_N = \sum_{n=1}^{N} a_n,$$

and

$$T_K = \sum_{k=0}^{K} 2^k a_{2^k}.$$

Show that

$$2S_{2^{k+1}} \geq 2S_{2^k} + 2^{k+1} a_{2^{k+1}},$$

and

$$S_{2^{k+2}-1} \leq S_{2^{k+1}-1} + 2^{k+1} a_{2^{k+1}}.$$

2^*. Prove the root test formulated in class.

3^*. Give an example of a convergent series with a divergent rearrangement. (note that by the theorem we proved in class, it is impossible if a series is *absolutely* convergent!)

4. Let $x \in (-1, 1)$ and $q \in \mathbb{R}$. Show that $\sum_{n=1}^{\infty} n^q x^n$ is absolutely convergent.

5. Is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log_2 n}$ convergent? Is it absolutely convergent? Include proofs.