HOME WORK VI, DIFFERENTIAL GEOMETRY: RIEMANN CURVATURE; MINIMIZING PROPERTIES OF GEODESICS.

Due March 24. The Home Work must be uploaded on Canvas as a pdf. In addition to the class notes, you may find helpful the book by DoCarmo "Riemanian geometry", Chapters 3 and 4. Problems marked with an asterisk are optional. Please contact me if you have any questions!

1 (DoCarmo, question 4 in Chapter 4). Let M be a Riemannian manifold with the following property: given any two points p and q in M, the parallel transport from p to q does not depend on the curve that connects p and q. Prove that the manifold is flat, that is, for all $p' \in M$, and all vectors $X, Y, Z \in T_{p'}M$, one has $R(X, Y)Z = 0$.

2. Prove the fact stated in class that for every $p \in M$, there exists a number $\epsilon > 0$ such that the geodesic ball $B_\epsilon(p)$ is strongly convex.

3 (DoCarmo, question 9, Chapter 4). Prove that the scalar curvature $K(p)$ at $p \in M$, where M is an $n-$dimensional Riemannian manifold, is given by

$$K(p) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} Ric_p(\theta) d\theta,$$

where $|\mathbb{S}^{n-1}|$ is the total area of the sphere, and the integration is with respect to the surface measure on the sphere.

4^*. A Riemannian manifold is called Einstein manifold if for all X, Y in the tangent bundle of M, $Ric(X,Y) = \lambda g(X,Y)$, for some function $\lambda : M \to \mathbb{R}$.

a) Prove that for any connected Einstein manifold M of dimension at least 3, λ is a constant function.

b) Prove that any 3-dimensional connected Einstein manifold has constant sectional curvature.

c) Is every 3-dimensional connected manifold of constant sectional curvature also an Einstein manifold?