HOME WORK 7, ANALYSIS I

Due November 14. Problems marked with asterisk are optional, but highly recommended. Please contact me if you have any questions!

1. In class we showed that there is a bijection $f : (-1, 1) \to \mathbb{R}$. Please show that there is a bijection $g : [0, 1] \to \mathbb{R}$.

2. Give an example of a set which is simultaniously:
a) not closed,
b) its compliment in $\mathbb{R}$ is not closed,
c) not countable,
d) and is not a finite union of intervals.
Please prove each fact about this set.

3. Prove that every point of $[0, 1]$ is a limit point.

4*. For a set $K \subset \mathbb{R}$, we say that *from every covering of* K *by open intervals one may select a finite subcovering*, if the following holds: for every set $A \subset \mathbb{R}$, for every family of intervals $I_\alpha = (a_\alpha, b_\alpha)$, where $\alpha \in A$, such that

$$K \subset \cup_{\alpha \in A} I_\alpha,$$

there exists a **finite** set $N \subset A$ such that

$$K \subset \cup_{\alpha \in N} I_\alpha.$$

a) With $K = (0, 1)$, give an example of a covering (indexed by some infinite set A) such that one may not select a finite subcovering.
b) Prove that from every covering of K by open intervals one may select a finite subcovering if and only if K is a compact (i.e. if from every sequence in K one may select a subsequence converging to an element of K).

5. a) Give an example of a function on $[-1, 1]$ which has limit 5 at $x = 0$;

b) Give an example of a function on $[-1,1]$ which does not have a limit at $x=0$.

6^*. Give an example of a function on $[0,1]$ which is continuous at only one point (and is discontinuous at the rest).

7^*. Prove that for every non-compact set $S \subset \mathbb{R}$ there exists a continuous function $f : S \to \mathbb{R}$ which is unbounded.

8. Let $f : [0,1] \to [0,1]$ be a continuous function. Show that there exists an $x \in [0,1]$ such that $f(x) = x$.

Hint: how and to what function should you apply the intermediate value theorem?