HOME WORK I, HIGH-DIMENSIONAL GEOMETRY AND PROBABILITY, SPRING 2018

Due January 30. All the questions marked with an asterisk(s) are optional. The questions marked with a double asterisk are not only optional, but also have no due date.

Question 1. Let $F : \mathbb{R} \to \mathbb{R}$ be a real-valued function, and m be a positive number. We make the following assumptions.

- F attains the absolute maximum at the point s_0, and for every $s \neq s_0$ we have $F(s) < F(s_0)$.
- Further, assume that there exist numbers $a, b > 0$ such that $F(s) < F(s_0) - b$ whenever $|s - s_0| > a$.
- Suppose that the integral $\int e^{F(s)} ds < \infty$.
- Suppose that F is twice differentiable in some neighborhood of s_0.
- Suppose that $F''(s_0) < 0$.

Prove that when $m \to \infty$, the integral

$$\int e^{mF(s)} ds = (1 + o(1)) e^{mF(s_0)} \frac{\sqrt{2\pi}}{\sqrt{-mF''(s_0)}}.$$

Hint 1: Observe that WLOG $s_0 = F(s_0) = 0$, and that F is equal to $-\infty$ outside of the support.

Hint 2: Pick any $\epsilon > 0$ and note that one may find a $\delta > 0$ so that for all $s \in (-\delta, \delta)$ we have

$$|F(s) - \frac{F''(0)s^2}{2}| \leq \epsilon.$$

Hint 3: Find an estimate for

$$\int_{-\delta}^{\delta} e^{mF(s)} ds.$$

Hint 4: Note that the assumptions imply that for every $\delta > 0$ there is $\eta(\delta) > 0$ such that $F(s) < F(s_0) - \eta(\delta)$;

Hint 5: Find an estimate for $\int_{\delta}^{\infty} e^{mF(s)} ds$ and $\int_{-\infty}^{-\delta} e^{mF(s)} ds$; to do that, use the previous hint, and also note that $e^{mF(s)} = e^{(m-1)F(s)} e^{F(s)}$. Use the assumption about the converging integral as well.

Hint 6: Carefully make sure that the assumptions allow you to let $m \to \infty$ and $\epsilon \to 0$.

Question 2. All the questions below require an answer up to a multiplicative factor of $1 + o(1)$, when $n \to \infty$.

a) Find $\frac{|B_2^n|_n}{|B_2^{n-1}|_{n-1}}$.

Hint: Use the formula from Question 1 and the Fubbini theorem. Note that this method is alternative to the one we used in class to express $|B_2^n|_n$.

b) Find the volume of

$$\{x \in \mathbb{R}^n : |x| \leq 2, x_1 \in [a, b]\},$$

where b1) $a = 0$, $b = 0.1$; b2) $a = -\frac{1}{\sqrt{n}\log n}$, $b = \frac{1}{n}$.

Hint: Use the expression for $|B_2^k|_k$ which we derived in class.

c) Using any method you like, find the volume of

$$conv(\{x \in \mathbb{R}^n : \, |x| < 3, x_2 = 0\} \cup \{x \in \mathbb{R}^n : \, |x - e_2| < 1, x_2 = 1\}).$$

d) Let γ be the standard Gaussian measure on $\mathbb{R}^n$ with density $\frac{1}{\sqrt{2\pi}^n} e^{-\frac{|x|^2}{2}}$. For each $t \in (0, \infty)$, find $\gamma(\{x : |x| > t\})$, depending on t (find the best approximation you can for each range).

e) Let μ be the **probability** measure with density $C(n)e^{-|x|^3}$. Find $C(n)$.

f) Let μ be as above. Let $R \in (0, \infty)$ be such that $\mu(RB_2^n) = \frac{1}{2}$. Find R.

Question 3. Let A be a convex set in $\mathbb{R}^n$ satisfying $x_1 = 0$ for all $x \in A$. Find the volume of $conv(A, Re_1)$, in terms of $|A|_{n-1}$, R and n.

Question 4. Prove that for any convex **body** K in $\mathbb{R}^n$ and for any point $x \in \mathbb{R}^n \setminus K$, there exists a vector $\theta \in \mathbb{S}^{n-1}$ and a number $\rho \in \mathbb{R}$ such that $\langle x, \theta \rangle > \rho$ and for all $y \in K$, $\langle y, \theta \rangle < \rho$.

Question 5^***.** Prove that a convex hull of a finite number of points in $\mathbb{R}^n$ either has an empty interior, or can be expressed as an intersection of a finite number of half spaces.

Question 6^{**}**.** Find a function $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every symmetric convex body K in $\mathbb{R}^n$ with $|K|_n = 1$, there exists a vector $u \in \mathbb{S}^{n-1}$ (possibly depending on the body), such that $|K \cap u^\perp|_{n-1} \geq F(n)$.

Acceptable answers could be $F(t) = 20t^{-t}$, $F(t) = 5^{-t}$, $F(t) = 3t^{-2}$, $F(t) = \frac{1}{t}$, $F(t) = \frac{10}{\sqrt{t}}$, $F(t) = 100t^{-\frac{1}{4}}$, $F(t) = \frac{1}{\log t}$, $F(t) = 0.00001$, $F(t) = \sqrt{2}$, etc.