HOME WORK II, HIGH-DIMENSIONAL GEOMETRY AND PROBABILITY, SPRING 2018

Due February 13. All the questions marked with an asterisk(s) are optional. The questions marked with a double asterisk are not only optional, but also have no due date.

Question 1. a) Prove that for convex bodies K and L with non-empty interior,

$$h_{K+L}(x) = h_K(x) + h_L(x),$$

for all $x \in \mathbb{R}^n$.

$b)^*$ Prove that when h is a support function of a strictly convex compact region K in $\mathbb{R}^2$, the surface area measure has a density expressible in the form

$$f_K(u) = h(u) + \ddot{h}(u),$$

for all $u \in \mathbb{S}^1$. Note that $h + \ddot{h}$ is translation invariant.

Question 2. In this question, K and L stand for convex bodies in $\mathbb{R}^n$ with non-empty interior, containing the origin.

a) Prove that $K^{oo} = K$.

b) Prove that for a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ with $\det T \neq 0$,

$$(T^t K)^o = T^{-1} K^o.$$

Conclude that a polar of an ellipsoid is an ellipsoid.

c) Prove that

$$(B_p^n)^o = B_q^n,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, for all $p, q > 1$.

d) Prove that

$$(K \cap L)^o = conv(K^o \cup L^o).$$

e) Prove that for every subspace H of $\mathbb{R}^n$

$$(K|H)^o \cap H = K^o \cap H.$$

f) Prove that if $K \subset L$, one has $L^o \subset K^o$.

g) Prove that if K is symmetric then K^o is symmetric.

Question 3. Let P be a polytope given by

$$P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq a_i, \forall i = 1, ..., N\},$$

for some unit vectors $u_1, ..., u_N$ and positive numbers $a_1, ..., a_N$, and suppose that P is bounded. Show that

$$P^o = c\bar{onv}\{\frac{u_1}{a_1}, ..., \frac{u_N}{a_N}\}.$$

Question 4. Below S_u stands for Steiner symmetrization with respect to $u^\perp$; K stands for a convex body in $\mathbb{R}^n$ with non-empty interior, containing the origin. Show that

a) $S_u(aK) = aS_uK$ for all $a > 0$.

b) Recall that for a compact set $A \subset \mathbb{R}^n$, the diameter $diam(A) = max_{x,y \in A}|x - y|$. Prove that

$$diam(S_u(K)) \leq diam(K).$$

Question 5^***.** Verify Mahler's conjecture in $\mathbb{R}^2$ for symmetric polygons: show that for any symmetric polygon P in $\mathbb{R}^2$,

$$|P| \cdot |P^o| \geq 8.$$

Question 6^{**}**.** a) Check that the Log-Brunn-Minkowski inequality in $\mathbb{R}^2$ implies that for every pair of **even**, $[-\pi, \pi]-$periodic infinitely smooth functions ψ and h, such that $h + \ddot{h} > 0$ and $h > 0$, one has

$$\left(\int_{-\pi}^{\pi} (h^2 - \dot{h}^2) du\right) \left(\int_{-\pi}^{\pi} (\psi^2 - \dot{\psi}^2 + \psi^2 \frac{h + \ddot{h}}{h}) du\right) \leq 2 \left(\int_{-\pi}^{\pi} (h\psi - \dot{h}\dot{\psi}) du\right)^2. \tag{0.1}$$

Hint: Use similar reasoning to the one we used to derive the analogous corollary of Brunn-Minkowski inequality; instead of K_s with support function $h_s = h + s\psi$, consider $h_s = h\psi^s$.

b) Prove (0.1) directly (without appealing to the validity of Log-Brunn-Minkowski inequality on the plane.)