HOME WORK III, HIGH-DIMENSIONAL GEOMETRY AND PROBABILITY, SPRING 2018

Due February 27. All the questions marked with an asterisk(s) are optional. The questions marked with a double asterisk are not only optional, but also have no due date.

Question 1^***.** a) Show that the minimal number of directions necessary to illuminate the boundary of a convex **body** K, the minimal number of points necessary to illuminate ∂K, and the minimal number of translates of the interior of K which contain K in their union, are equal.
b) Show that if the convex set K is not compact, some of the previous assertions fail.

Question 2. Generalize Borell's lemma to log-concave distributions: namely, prove that for any log-concave measure μ, any convex set K, any symmetric convex set A such that $\frac{\mu(K\cap A)}{\mu(K)} = v > \frac{1}{2}$, for all $t > 1$, one has

$$\mu(K \cap (tA)^c) \leq \mu(K) \cdot v \cdot \left(\frac{1-v}{v}\right)^{\frac{t+1}{2}}.$$

Question 3. Prove (as a corollary of Prekopa's inequality) that if μ is log concave, then for all subspaces E, the marginal measure $\pi_E(\mu)$ is also log-concave.

Question 4. Let $p \geq -\frac{1}{n}$, and suppose functions f, g and h on $\mathbb{R}^n$ satisfy

$$h(\lambda x + (1-\lambda)y) \geq M_p^\lambda(f(x), g(y)).$$

Show that then

$$\int h \geq M^\lambda_{\frac{p}{np+1}}(\int f, \int g).$$

Question 5. For the shadow system $K_t = conv\{x + \alpha(x)v \;:\; x \in A\}$, define the convex body

$$\tilde{K} = conv\{x + \alpha(x)e_{n+1}\} \subset \mathbb{R}^{n+1}.$$

Show that for $u \in e_{n+1}^{\perp}$,

$$h_{K_t}(u) = h_{\tilde{K}}(u + t\langle u, v\rangle e_{n+1}).$$

Question 6^***.** Prove the Blascke-Santalo inequality using directly Steiner symmetrization in place of Shadow systems.

Hint: Use the Brunn-Minkowski inequality in place of Borell's theorem.