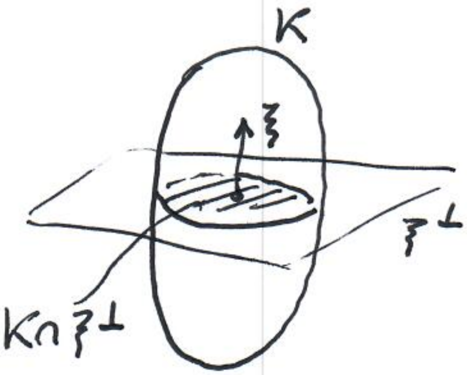
Busemann - Petty problem.

1956: K, L origin-symmetric convex bodies in $\mathbb{R}^n$

$$|K \cap \xi^\perp| \le |L \cap \xi^\perp| , \quad \forall \xi \in S^{n-1}$$

Does it necessarily follow that

$$|K| \le |L| \ ?$$

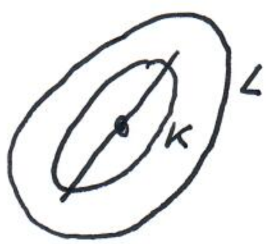


$\xi^\perp = \{x \in \mathbb{R}^n : (x, \xi) = 0\}$

$|K|$ volume of proper dim

$n = 2$

$K \subset L$



Busemann (1960): Yes, if K is an ellipsoid

Hadwiger (1968): Yes, if K, L are solids of revolution in dim 3

Larman, Rogers (1975): No, $n \geq 12$

Ball (1986): No, $n \geq 10$

Giannopoulos (1990), Bourgain (1991): No, $n \geq 7$

Papadimitrakis (1992), Gardner (1994): No, $n \geq 5$

Gardner (1994): Yes, $n = 3$

Used the connection with intersection bodies found by Lutwak (1988):

The answer to BP in $\mathbb{R}^n$ is affirmative

$\Longleftrightarrow$ every origin-symmetric convex body in $\mathbb{R}^n$ is an intersection body

Zhang (1999): Yes, if $n = 4$

Gardner, K., Schlumprecht (1999): unified solution in all dimensions

Yes, $n \leq 4$, No, $n \geq 5$

4 papers in Ann. Math.

Intersection bodies.

Spherical Radon transform:

$R: C(S^{n-1}) \longrightarrow C(S^{n-1})$

$$Rf(\xi) = \int_{S^{n-1}\cap \xi^\perp} f(x)\,dx, \quad f \in C(S^{n-1}),\ \xi \in S^{n-1}$$

μ measure on S^{n-1},

$$\langle R\mu, f\rangle = \langle \mu, Rf\rangle = \int_{S^{n-1}} Rf(x)\,d\mu(x)$$

$\|x\|_K = \min\{a \geq 0 : x \in aK\}$

$r_K(x) = \|x\|_K^{-1}, \ x \in S^{n-1}$

Definition (Lutwak (1988)):

A star body K in $\mathbb{R}^n$ is called an intersection body if $\exists$ measure ν_K on S^{n-1} so that

$\|x\|_K^{-1} = R\nu_K$ as functionals on $C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x)\,dx = \langle R\nu_K, f\rangle = \int_{S^{n-1}} Rf(x)\,d\nu_K(x)$$

$$\forall f \in C(S^{n-1}).$$

K. (1998): An origin-symmetric star body K in $\mathbb{R}^n$ is an intersection body $\iff$ $\|x\|_K^{-1}$ is a positive definite distribution

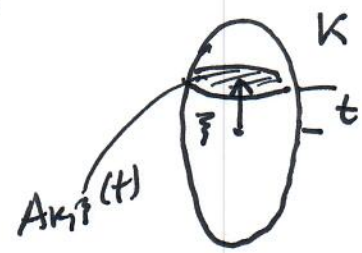
i.e. $\langle (\|x\|_K^{-1})^\wedge, \varphi \rangle \geq 0$ $\forall \varphi \geq 0$, Schwartz test function

(i) the unit ball of any finite dimensional subspace of L_p, $0<p\leq 2$ is an intersection body

(ii) the unit ball of ℓ_p^n, $2<p\leq +\infty$ is an intersection body only if $n\leq 4$.

$A_{K,\xi}(t) = |K \cap \{\xi^\perp + t\xi\}|$, $t\in\mathbb{R}$

parallel section function of K in the direction ξ



Gardner, K., Schlumprecht (1999):

$$A_{K,\xi}^{(m)}(0) = \frac{(-1)^{m/2}}{\pi(n-m-1)} \left(\|x\|_K^{-n+m+1}\right)^\wedge(\xi)$$

if K is infinitely smooth star body in $\mathbb{R}^n$, m even integer, $m\geq 0$.

$n=4$, $m=2$: $A''_{K,\xi}(0)\leq 0$ by Brunn's Th.

$n=5$, $m=3$: A''' is not controlled by convexity if K- origin-sym. convex

Isomorphic BP:

Does there exist $C>0$ so that $\forall n$, $\forall$ origin-symmetric convex K, L in $\mathbb{R}^n$

$$|K\cap\xi^{\perp}|\leq|L\cap\xi^{\perp}|,\ \forall\xi\in S^{n-1}$$

imply

$$|K|\leq C|L| \ ?$$

Slicing problem.

Does there exist $C>0$ so that $\forall n$, $\forall$ origin-symmetric convex K in $\mathbb{R}^n$, $|K|=1$ $\exists$ hyperplane section of K with area $\geq c$.

$\exists C>0$: $\forall n$, $\forall$ origin-symm. convex K in $\mathbb{R}^n$

$$\boxed{|K|^{\frac{n-1}{n}}\leq C\max_{\xi\in S^{n-1}}|K\cap\xi^{\perp}|} \ ?$$

Best-to-date estimate $C\leq O(n^{1/4})$ by Klartag who removed a log term from an earlier result of Bourgain.

IBP $\Leftrightarrow$ SP

IBP $\Rightarrow$ SP; $L=cB_2^n$, where

$$c=\left(\frac{\max_{\xi}|K\cap\xi^{\perp}|}{|B_2^{n-1}|}\right)^{\frac{1}{n-1}}$$

$\forall\theta\in S^{n-1}$, $|cB_2^n\cap\theta^{\perp}|=c^{n-1}\cdot|B_2^{n-1}|=\max_{\xi}|K\cap\xi^{\perp}|\geq|K\cap\theta^{\perp}|$

$$\underset{\text{by IBP}}{\Rightarrow}\quad |K|^{\frac{n-1}{n}}\leq C^{\frac{n-1}{n}}\cdot|L|^{\frac{n-1}{n}}=C^{\frac{n-1}{n}}\underbrace{\frac{|B_2^n|^{\frac{n-1}{n}}}{|B_2^{n-1}|}}_{C_1}\max_{\xi}|K\cap\xi^{\perp}|$$

Slicing inequalities for functions.

Zvavitch (2005): f even continuous strictly positive function on $\mathbb{R}^n$, K, L origin-symm. convex in $\mathbb{R}^n$

$$\int_{K\cap\xi^\perp} f \le \int_{L\cap\xi^\perp} f, \quad \forall \xi\in S^{n-1}$$

BP problem for arbitrary function

$$\Longrightarrow \int_K f \le \int_L f \quad ?$$

Yes if $n\le 4$, No if $n\ge 5$.

Isomorphic BP for functions

$\mathcal{I}_n$ the class of intersection bodies in $\mathbb{R}^n$.

The Banach-Mazur distance

$$d_{BM}(K, \mathcal{I}_n) = \inf\{a\ge 0 : \exists \mathcal{D}\in \mathcal{I}_n : \mathcal{D}\subset K \subset a\mathcal{D}\}$$

K., Zvavitch (2015): K, L star bodies in $\mathbb{R}^n$, f locally integrable on $\mathbb{R}^n$

$$\int_{K\cap\xi^\perp} f \le \int_{L\cap\xi^\perp} f, \ \forall\xi \Longrightarrow \boxed{\int_K f \le d_{BM}(K, \mathcal{I}_n)\cdot \int_L f.}$$

If K origin-symm. convex, then by John's Th

$$\boxed{\int_K f \le \sqrt{n} \int_L f.}$$

Slicing inequality for functions.

K star body in $\mathbb{R}^n$, $|K|=1$

f integrable on K, $\int_K f = 1$, $f \geq 0$

Does there exist $C>0$: $\forall K$, $\forall f$

$$\exists\, \xi \in S^{n-1}: \quad \int_{K\cap\xi^\perp} f \geq c.$$

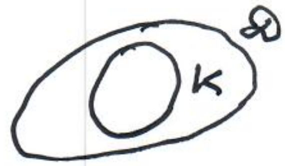
Equivalently, find the minimal constant C_n :

$\forall$ star body in $\mathbb{R}^n$, $\forall f \geq 0$ integrable on K

$$\boxed{\int_K f \leq C_n \cdot \max_\xi \int_{K\cap\xi^\perp} f \cdot |K|^{1/n}}$$

The outer volume ratio distance:

$$d_{ovr}(K, \mathcal{I}_n) = \inf \left\{ \left(\frac{|D|}{|K|}\right)^{1/n} : K \subset D,\ D \in \mathcal{I}_n \right\}$$



K. (2015): $\forall$ star body K in $\mathbb{R}^n$,

$\forall f \geq 0$ integrable on K

$$\boxed{\int_K f \leq 2\, d_{ovr}(K, \mathcal{I}_n) \cdot \max_\xi \int_{K\cap\xi^\perp} f \cdot |K|^{1/n}}$$

(i) Does not follow directly from IBP for functions

(ii) $d_{ovr} < d_{BM}$

(i) If K symm. convex, by John's Th.

$$d_{ovr}(K, I_n) \le \sqrt{n},$$

because I_n contains ellipsoids

$$\int_K f \le 2\sqrt{n} \cdot \max_\xi \int_{K \cap \xi^\perp} f \cdot |K|^{1/n}$$

does not depend on K, f

For non-symmetric K, proved later by Chasapis, Giannopoulos, Liakopoulos (2017)

(ii) K unconditional convex

$$d_{ovr}(K, I_n) \le e$$

(iii) For unit balls of subspaces of L_p, $p>2$

$d_{ovr}(K, I_n) \le C\sqrt{p}$ E. Milman (2005)

(iv) Obviously, for intersection bodies

$d_{ovr}(K, I_n) = 1$

This includes unit balls of subspaces of L_p, $0<p\le 2$

(v) Klartag, K. (2018) $\exists$ symm. convex M in $\mathbb{R}^n$, and a probability density f on M such that

$$\int_{M \cap \xi^\perp} f \le \frac{C\sqrt{\log\log n}}{\sqrt{n}} \cdot |M|^{-1/n}$$

$\sqrt{\log\log n}$ was removed by Klartag, Livshyts (2019+)

So the constant $\sqrt{n}$ is optimal in the slicing inequality for functions.

K., Paouris, Zvavitch (2020+):

K, L star bodies in $\mathbb{R}^n$, $f, g \geq 0$ locally int. in $\mathbb{R}^n$, $\|g\|_\infty = g(0) = 1$.

Suppose that

$$\int_{K\cap\xi^\perp} f \leq \int_{L\cap\xi^\perp} g, \quad \forall \xi \in S^{n-1}. \text{ Then}$$

$$\boxed{\int_K f \leq d_{ovr}(K, \mathcal{I}_n) \cdot \frac{n}{n-1} |K|^{1/n} \left(\int_L g\right)^{\frac{n-1}{n}}}$$

Corollary. Let $g \equiv 1$.

If $\int_{K\cap\xi^\perp} f \leq |L\cap\xi^\perp|$, $\forall \xi$ then

$$\boxed{\int_K f \leq d_{ovr}(K, \mathcal{I}_n) \cdot \frac{n}{n-1} |L|^{\frac{n-1}{n}} \cdot |K|^{1/n}}$$

Put $L = cB_2^n$,

$$c = \left(\frac{\max_\xi \int_{K\cap\xi^\perp} f}{|B_2^{n-1}|}\right)^{\frac{1}{n-1}}$$

$$\int_{K\cap\theta^\perp} f \leq \max_\xi \int_{K\cap\xi^\perp} f = |cB_2^n \cap \theta^\perp|. \text{ So}$$

$$\int_K f \leq \frac{n}{n-1} d_{ovr}(K, \mathcal{I}_n) \cdot |K|^{1/n} \cdot \frac{|B_2^n|^{\frac{n-1}{n}}}{|B_2^{n-1}|} \cdot \max_\xi \int_{K\cap\xi^\perp} f$$

Proof.

$$\int_{K\cap\xi^\perp} f = \int_{S^{n-1}\cap\xi^\perp} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-2} f(r\theta)\, dr \right) d\theta =$$

$$= R\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-2} f(r\,\cdot)\, dr \right)(\xi)$$

The condition of Theorem:

$$(*) \quad R\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-2} f(r\,\cdot)\, dr \right)(\xi) \le R\left(\int_0^{\|\cdot\|_L^{-1}} r^{n-2} f(r\,\cdot)\, dr \right)(\xi), \quad \forall \xi \in S^{n-1}$$

For a small $\delta > 0$, let $D \in I_n$ be a body such that

$K \subset D$ and

$$|D|^{1/n} \le (1+\delta)\, d(K, I_n) \cdot |K|^{1/n}$$

Let ν_D be the measure on S^{n-1} corresponding to D by definition of intersection body.

Integrating both sides of $(*)$ by $d\nu_D(\xi)$:

$$\int_{S^{n-1}} \|x\|_K^{-1} \left(\int_0^{\|x\|_K^{-1}} r^{n-2} f(rx)\, dr \right) dx \le$$

$$\le \int_{S^{n-1}} \|x\|_L^{-1} \left(\int_0^{\|x\|_L^{-1}} r^{n-2} g(r\theta)\, dr \right) dx.$$

$$\Longleftrightarrow \quad \int_K \|x\|_D^{-1} f(x)\, dx \le \int_L \|x\|_L^{-1} g(x)\, dx$$

Since $K \subset D$, we have $1 \ge \|x\|_K \ge \|x\|_D \Rightarrow$

$$\Rightarrow \int_K \|x\|_D^{-1} f(x)\, dx \ge \int_K \|x\|_K^{-1} f(x)\, dx \ge \int_K f$$

By V. Milman - Pajor, Lemma 2.1

$$\left(\frac{\int_L \|x\|_D^{-1} g(x)\, dx}{\int_D \|x\|_D^{-1}\, dx} \right)^{\frac{1}{n-1}} \leq \left(\frac{\int_L g}{\int_D dx} \right)^{\frac{1}{n}}$$

Here we use $g(0) = \|g\|_\infty = 1$

Since $\int_D \|x\|_D^{-1} dx = \frac{n}{n-1} |D|$, we get

$$\int_L \|x\|_D^{-1} g(x)\, dx \leq \frac{n}{n-1} \left(\int_L g \right)^{\frac{n-1}{n}} \cdot |D|^{1/n} \leq$$

$$\leq (1+\delta)\, d_{ovr}(K, I_n) \frac{n}{n-1} \left(\int_L g \right)^{\frac{n-1}{n}} \cdot |K|^{1/n}, \quad \forall \delta > 0$$

Send $\delta \to 0$. ■

$g : \mathbb{R}^n \to \mathbb{R}^+$ measurable, $\|f\|_\infty = 1$,
K - convex body in $\mathbb{R}^n$

$$F(p) = \left(\frac{\int_{\mathbb{R}^n} \|x\|_K^p f(x)\, dx}{\int_K \|x\|_K^p\, dx} \right)^{\frac{1}{n+p}}$$

$\nearrow$ of p on $(-n, \infty)$

BP problems for moments of functions.

Bobkov, Klartag, K. (2018):

Let K, M be origin-symmetric star bodies in $\mathbb{R}^n$, and let $f \geq 0$ be an even continuous function on $\mathbb{R}^n$. Given $p \geq 1$, suppose that $\forall \xi \in S^{n-1}$

$$\int_K |(x,\xi)|^p f(x)\, dx \leq \int_M |(x,\xi)|^p f(x)\, dx .$$

Then

$$\int_K f \leq d_{BM}^p (M, L_p^n) \cdot \int_M f ,$$

where $d_{BM}(M, L_p^n)$ is the Banach-Mazur distance from M to the unit balls of n-dim subspaces of L_p

$(\mathbb{R}, \|\cdot\|)$ embeds isometrically in L_p iff $\exists \mu$ on S^{n-1}:

$$\|x\|^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu(\xi).$$

L_p^n is the class of unit balls of n-dim spaces that embed in L_p.

K., Paouris, Zvavitch (2020+):

Let K, M be star bodies in $\mathbb{R}^n$, $p>0$, f, g non-negative locally integrable functions on $\mathbb{R}^n$, $\|g\|_\infty = g(0)=1$.

Suppose $\forall \xi \in S^{n-1}$

$$\int_K |(x,\xi)|^p g(x)\,dx \le \int_M |(x,\xi)|^p f(x)\,dx.$$

Then

$$\left(\int_K g\right)^{\frac{n+p}{n}} \le \frac{n+p}{n}\, d_{ovr}^p(M, L_p^n)\cdot |M|^{p/n} \cdot \int_M f$$

Bobkov, Klartag, K. (2018):
$f \ge 0$ integrable function on a compact set $K \subset \mathbb{R}^n$, $p>2$. Then

$$\int_K f \le C\sqrt{p}\; d_{ovr}(K, L_p^n)\cdot |K|^{1/n} \max_H \int_{K\cap H} f$$

where C is an absolute constant, and max is taken over all affine hyperplanes in $\mathbb{R}^n$.

Distance inequalities

Klartag, K. (2018):

$$c\frac{\sqrt{n}}{\cancel{\sqrt{\log\log n}}} \le \sup_{\substack{K \text{ convex} \\ \text{symm}}} d_{ovr}(K, I_n) \le \sqrt{n}$$

log term removed by Klartag, Livshyts (2020+)

Bobkov, Klartag, K. (2019):

$$c\frac{\sqrt{n}}{\sqrt{p}\,\cancel{\sqrt{\log\log n}}} \le \sup_{K} d_{ovr}(K, L_p^n) \le \sqrt{n}$$

log term removed by Klartag, Livshyts

K., Paouris, Zvavitch (2020+):

$$d_{ovr}(K, L_p^n) \le c\sqrt{\frac{n+p}{p}}.$$ - sharp

Remarks and open questions.

1. Lower dim BP problem.

$$|K \cap H|_{n-k} \leq |L \cap H|_{n-k}, \ \forall H \in G_{n-k}$$

$$\Rightarrow |K| \leq |L| \ ?$$

Bourgain - Zhang (1999): No, if $n-k > 3$.

Open for two- and three dim sections.

2. BP problem for complex convex bodies.

Solved - K., König, Zymonopoulou (2008):

Yes in $\mathbb{C}^n$, $n \leq 3$, No, $n \geq 4$.

3. BP problem in hyperbolic and spherical spaces.

Solved. Yaskin (2006):

Spherical: Yes $n \leq 4$, No, $n \geq 5$

Hyperbolic: Yes, $n = 2$ No, $n \geq 3$

4. Non-symmetric case.

Sections through two interior points, No, $n \geq 2$ - Shane (2008)

Sections through a boundary point

Yes $n = 2$, No, $n \geq 3$

Formulate the problem so that

the answer is affirmative at least for $n=3$

5. BP problem for surface area.

$$S(K \cap \xi^{\perp}) \leq S(L \cap \xi^{\perp}), \ \forall \xi \in S^{n-1}$$

$$\Longrightarrow \ S(K) \leq S(L) \ ?$$

König, K. (2019): No, $n \geq 14$.