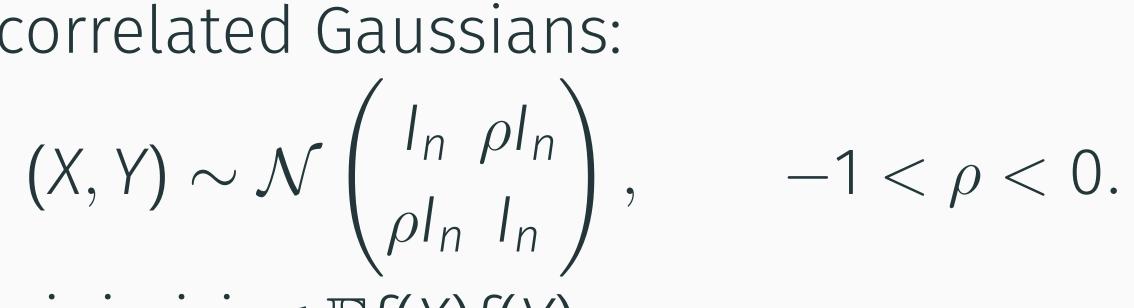
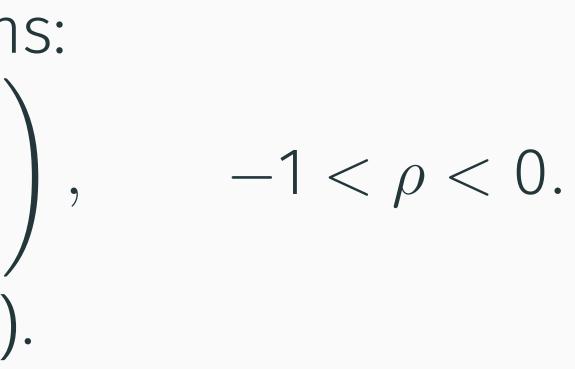
Vector-valued noise stability

Yeongwoo Hwang, Joe Neeman, Ojas Parekh, Kevin Thompson, John Wright

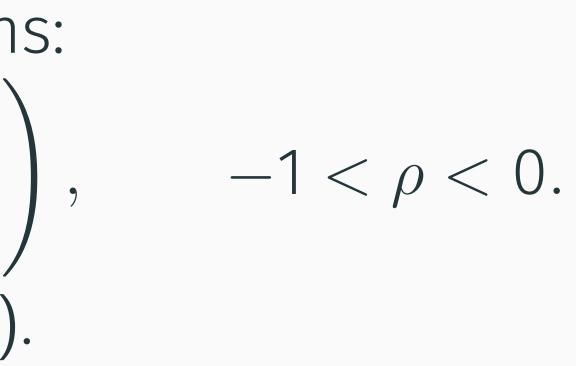
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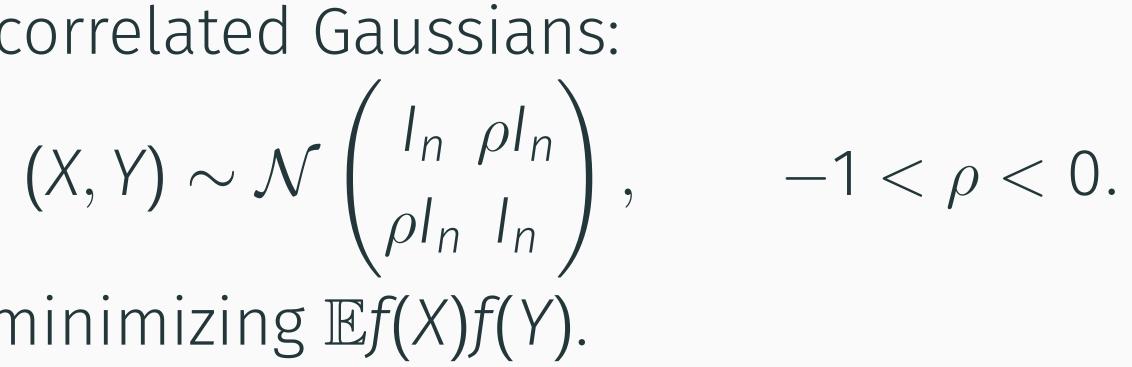
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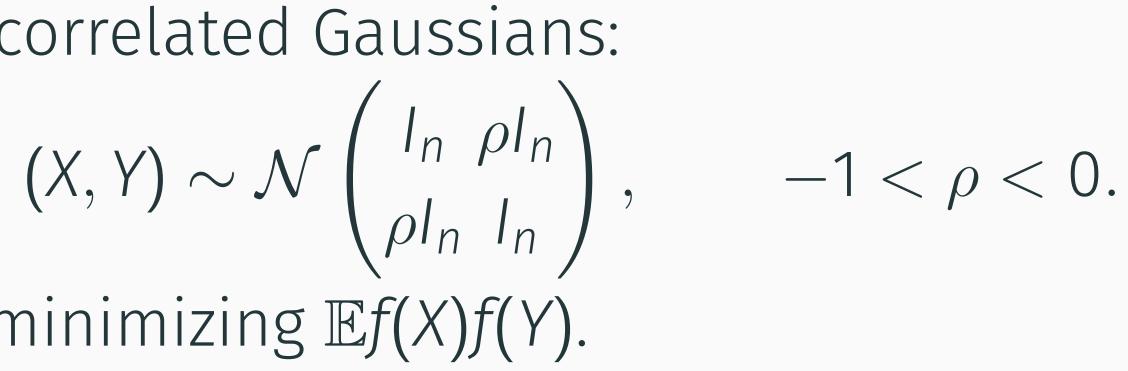
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GW has approximation ratio $\inf_{-1<\rho<0} \frac{1-\frac{2}{\pi} \operatorname{arcsin}(\rho)}{1-\rho} \approx 0.878.$

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Effect of rounding: $\rho := \langle y_i, y_j \rangle \Rightarrow |y_i - y_j$
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Theorem (Khot-Kindler-Mossel-O'Donnell, Mossel-O'Donnell-Oleszkiewicz) Doing better than 0.878 is Unique-Games-Hard.

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- 3. Spectral analysis: Solve it for $f: S^{k-1} \to B_2^k$.

Spectral argument

Let $g: [-1,1] \to \mathbb{R}$ be decreasing. $f: S^{k-1} \to \mathbb{R}^k$. $U_g f(u) := \int_{S^{k-1}} f(v) g(\langle u, v \rangle) d\sigma^{k-1}(v).$ $EfU_{q}f = E\langle f(X), f(Y) \rangle$

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Proof. Shur's lemma implies that eigenfunctions are spherical harmonics, so need to compute $\frac{U_g f(v)}{f(v)}$ for spherical harmonics f. Gegenbauer polynomials + NIST Handbook.

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Lemma If $D_{\theta}e(0) = 0$ then $D^2_{\theta,\theta}J(0) \leq 0$. Inequality is strict unless $f_{\alpha\theta} = f$ for all $\alpha \in \mathbb{R}$. Corollary If f is optimal and $D_{\theta}e(0) = 0$ then $f_{\alpha\theta} = f$ for all $\alpha \in \mathbb{R}$.

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Open problems

Conjecture If $0 < \rho < 1$ then $\mathbb{E}\langle f(X), f(Y) \rangle$ is maximized, among functions with $\mathbb{E}f = 0$, by $f(x) = \frac{(x_1, \dots, x_k)}{\sqrt{x_1^2 + \dots + x_k^2}}.$

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Question What about if $\mathbb{E}f = \mu \in B_2^k$?

Thank you!

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