Stability and Equality Case in the B-theorem

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Based on joint work with
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Notations

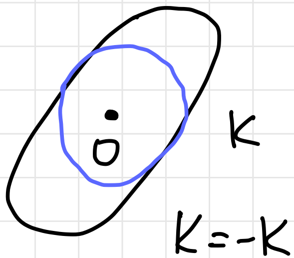
γ - Gaussian measure w/ density $\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|x|^2}{2}}$

$K \subset \mathbb{R}^n$ is symmetric $K = -K$

$aK = \{ax : x \in K\}$, $a > 0$ $a \in \mathbb{R}$

$r(K) = \max_r \{r > 0 \mid rB_2^n \subset K\}$

B_2^n - Eucl. Ball cent. at the origin



Prekopa (1971) - Leindler (1972)

$$\gamma\left(\frac{a+b}{2}K\right) \geq \sqrt{\gamma(aK)\gamma(bK)}$$

K - convex, no symmetry assumpt.

Latala (2002) (Banaszczyk conj)
proved by Cordero-Erausquin, Fradelizi, Maurey (2004)

$\boxed{\text{B-theorem}}$ convex $K = -K \subset \mathbb{R}^n$ $\forall a, b > 0$

$$\gamma(\sqrt{ab}\,K) \geq \sqrt{\gamma(aK)\gamma(bK)}$$

i.e. $\gamma(e^t K)$ is log-concave in t
when K sym. convex

a generalization: $x \in \mathbb{R}^n$

$$e^x K = \{(e^{x_1} y_1, \ldots, e^{x_n} y_n) \mid (y_1, \ldots, y_n) \in K\}$$

$\boxed{\text{Strong B-theorem}}$ $\forall x, y \in \mathbb{R}^n$

$$\gamma(e^{\frac{x+y}{2}} K) \geq \sqrt{\gamma(e^x K)\, \gamma(e^y K)}$$

Nayar, Tkocz (2014): $0 \in K$ is not enough

$K = -K$ - required

Cordero-Erausquin, Fradelizi, Maurey

? what other measures sat.

$$\gamma(\sqrt{ab}\,K) \geq \sqrt{\gamma(aK)\gamma(bK)} \qquad (1)$$

$$\gamma(e^{\frac{x+y}{2}}K) \geq \sqrt{\gamma(e^{x}K)\gamma(e^{y}K)}$$

Boroczky, Lutwak, Yang, Zhang (2012)
Saroglou (2016)

$\}$ $n=2$ $\Downarrow$ all even log-concave m. (1)

Bar-On (2014) uniform m. on convex bodies

Eskenazis, Nayar, Tkocz (2018) Gaussian mixtures

Cordero-Erausquin, Rotem (2021+) all-rotation log-concave m.

$\boxed{\underline{\text{Thm 1}}^{*}\ \text{(H, Livshyts, Rotem, Volberg) (2023+)}}$

$0 \le a < b < \infty$, convex $K = -K \subset \mathbb{R}^n$

Let $\sigma(\sqrt{ab}\,K) \le \sqrt{\sigma(aK)\,\sigma(bK)}\,(1+\varepsilon)$, $\varepsilon > 0$

Then
$$r(K) \ge \frac{1}{b}\sqrt{\log\left(\frac{c\log\left(\frac{b}{a}\right)^2}{n^2\varepsilon}\right)}$$ sharp

or
$$r(K) \le \frac{C\sqrt{n}}{a}\,\varepsilon^{\frac{1}{n+1}}\left(\log\frac{b}{a}\right)^{-\frac{2}{n+1}}$$

$\boxed{\text{Cor}}$

$$\gamma(\sqrt{\alpha\beta}\,K) = \sqrt{\gamma(\alpha K)\gamma(\beta K)}$$

$r(K) = \infty$ OR $r(K) = 0$

$K = \mathbb{R}^n$

K has an empty interior

for the strong B-thm:

Thm^* (2023+)

fix $\delta, \alpha, \beta > 0$, $x, y \in \mathbb{R}^n$ s.t. $|e^x| \leq |e^y|$

$K = -K$ convex

Let $\gamma(e^{\frac{x+y}{2}}K) \leq \sqrt{\gamma(e^x K)\gamma(e^y K)}\,(1+\varepsilon)$

for small enough $\varepsilon > 0$

Consider

$$\sigma^\delta = \{ i \in [n] \mid |x_i - y_i| \geq \delta \}$$

and let

$$\Omega_{\delta,\alpha}(K) = \{ x \in \partial K \mid \sum_{i \in \sigma^\delta} (n_x^i)^2 \geq \alpha \}$$

Then

there exists a vector $z \in [x, y]$ s.t.

$$\gamma^+(\Omega_{\delta,\alpha}(e^z K)) \leq \beta \gamma^+(\partial(e^z K))$$

OR

$$r(K) \geq |e^y|^{-1} \sqrt{\log \frac{\delta^2 \alpha \beta}{\varepsilon n^2}}$$

OR

$$r(K) \leq C |e^x|^{-1} \sqrt{n}\, \varepsilon^{\frac{1}{n+1}} (\delta^2 \alpha \beta)^{-\frac{1}{n+1}}$$

Notations and Properties

cdf $\Phi(x) = \gamma((-\infty, x]) \qquad x \in \mathbb{R}$

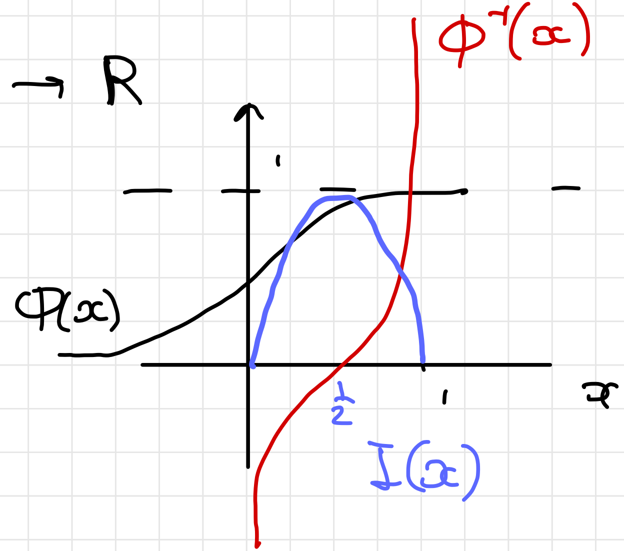
$\Phi^{-1} : [0,1] \to \mathbb{R}$

Isoperimetric profile $I : [0,1] \to \mathbb{R}$

$$I(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\Phi^{-1}(x)^2}{2}}$$

Gaus. isoper. ineq.

$$\gamma^+(\partial K) \geq I(\gamma(K))$$



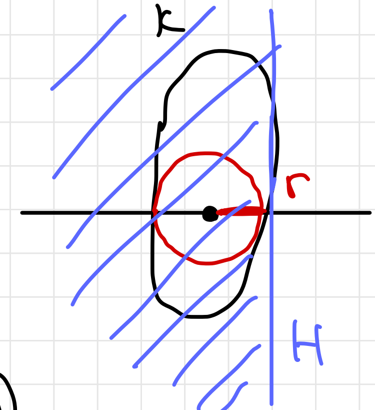
concave, max at $x = \frac{1}{2}$

1) Lower Bound on $\gamma^+(\partial K)$ in terms of r

Lemma*

convex $K = -K \subset \mathbb{R}^n$

$r > 0$



if $\gamma(K) \geq \frac{1}{2} \Rightarrow$

$$\gamma^+(\partial K) \geq I(\gamma(K))$$

$$\geq I(\gamma(H)) = I(\Phi(r))$$

$$\Rightarrow \gamma^+(\partial K) \geq \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}}$$

if $\gamma(K) \leq \frac{1}{2}$ then by concavity of I and Gaus. isop. ineq.

$$\gamma^+(\partial K) \geq \sqrt{\frac{2}{\pi}}\, \gamma(rB_2^n) \geq \left(\frac{cr}{\sqrt{n}}\right)^n e^{-\frac{r^2}{2}}$$

2) Rough estimates for Gaussian integrals over Balls

$\boxed{\text{Lemma}^*}$

$$\gamma(2\sqrt{n}\, B_2^n) \geq \frac{3}{4}, \quad \int_{2\sqrt{n}B_2^n} |x|^2 \, d\gamma \geq cn$$

$$\forall r > 0 \qquad \int_{rB_2^n} |x^2| \, d\gamma \geq \left(\frac{c}{\sqrt{n}}\right)^n r^{n+2} e^{-\frac{r^2}{2}}$$

the proof uses probabilistic bounds

3) Lemma* + Lemma* $\Rightarrow$ $\boxed{\text{Prop*}}$

$$\frac{\gamma(K)}{\int_{rB_2^n} |x|^2 \, d\sigma} + \frac{\gamma(K)}{r\gamma^+(\partial K)} \geq \frac{1}{\delta}, \quad \delta < c_0$$

$$\Rightarrow \quad r \geq \sqrt{\log \frac{1}{\delta}} \quad \text{OR} \quad r \leq C\sqrt{n}\,\delta^{\frac{1}{n+1}}$$

4) Stability of Gaussian Poincare ineq.

- for convex sets
- for quadratic func. („symmetric")

5) Stability + Thm 1 $\Rightarrow$ Thm 2

$\Downarrow$

$\boxed{\text{Corollary}}$

about equality case

$$\gamma\left(e^{\frac{x+y}{2}}K\right) = \sqrt{\gamma(e^{x}K)\,\gamma(e^{y}K)}$$

<u>Application</u> uniqueness of the Bobkov maximal Gaussian measure position for a convex Body K.

Thank you!