Mean-Value Inequalities for Convex Domains

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Georgia Tech, 2021

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Goal of the Talk

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There are relatively few results and lots of fun questions!

The Mean Value Theorem I

Let $B_r(0) \subset \mathbb{R}^d$ and let $\Delta f = 0$ for some $f : B_r(0) \to \mathbb{R}$. Then

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Proof.

Mean-Value Theorem and Maximum Principle. □

The Mean Value Theorem II

Let $B_r(0) \subset \mathbb{R}^d$ and let $\Delta f = 0$ for some $f : B_r(0) \to \mathbb{R}$. Then

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General Mean-Value Inequalities?

What if the domain is not a ball?

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How does the constant c_Ω depend on Ω?

Let's start with something 'simpler': convex functions.

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JENSEN-TYPE GEOMETRIC SHAPES

PAWEŁ PASTECZKA

ABSTRACT. We present both necessary and sufficient conditions to the convex closed shape X such that the inequality

$$\frac{1}{|X|}\int_X f(x)\,dx \le \frac{1}{|\partial X|}\int_{\partial X} f(x)\,dx$$

is valid for every convex function $f\colon X \to \mathbb{R}$ (∂X stands for the boundary of X).

It is proved that this inequality holds if X is (i) an n-dimensional parallelotope, (ii) an n-dimensional ball, (iii) a convex polytope having an inscribed sphere (tangent to all its facets) with center in the center of mass of ∂X.

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$$\int_0^1 f(x)dx \leq \frac{f(0)+f(1)}{2}$$

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or, for $\Omega = [0,1]$,

$$\frac{1}{|\Omega|}\int_\Omega f(x)dx \leq \frac{1}{|\partial\Omega|}\int_{\partial\Omega} f(x)dx.$$

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- Ω is a Platonic solids (Pasteczka)
- Ω is some type of polytope (Pasteczka)
- ...

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then Ω and $\partial\Omega$ have the same center of mass.

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Proof.

Plug in $f(x) = \langle a, x\rangle + b$.

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$$\frac{1}{|\Omega|}\int_{\Omega} f(x)dx = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} f(x)dx$$

for all functions of this type. □

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Conjecture (Pasteczka)

'Our conjecture is that every convex shape which satisfies this condition is of Jensen-type'.

I think this would be really nice if it were true (maybe too nice?)

Hermite-Hadamard Inequalities

Proposition (Pasteczka)

If, for all convex $f : \Omega \to \mathbb{R}$

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then Ω and $\partial\Omega$ have the same center of mass.

In particular, if Ω and $\partial\Omega$ have different centers of mass, then the optimal constant c_Ω

$$\frac{1}{|\Omega|} \int_{\Omega} f(x)dx \leq \frac{c_\Omega}{|\partial\Omega|} \int_{\partial\Omega} f(x)dx$$

satisfies $c_\Omega > 1$.

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$f : \Omega \to \mathbb{R}^d$ convex and

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Theorem (S. 2018)

$c_\Omega \leq c_n$ for all convex domains $\Omega \subset \mathbb{R}^n$.

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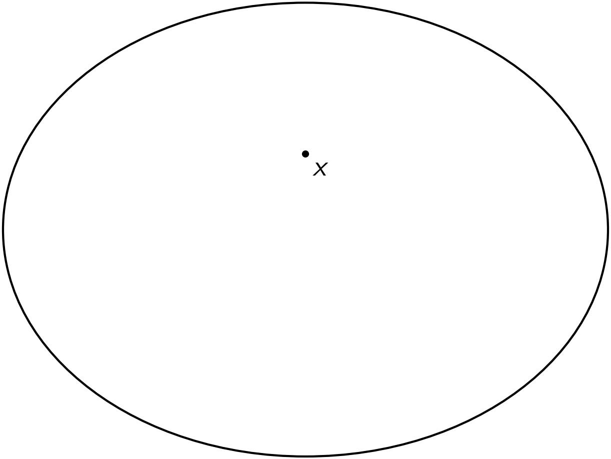
I will later show much better results. But what is interesting here is that there is a fun transport problem hiding here. I always thought that this was independently interesting.

A Transport Problem



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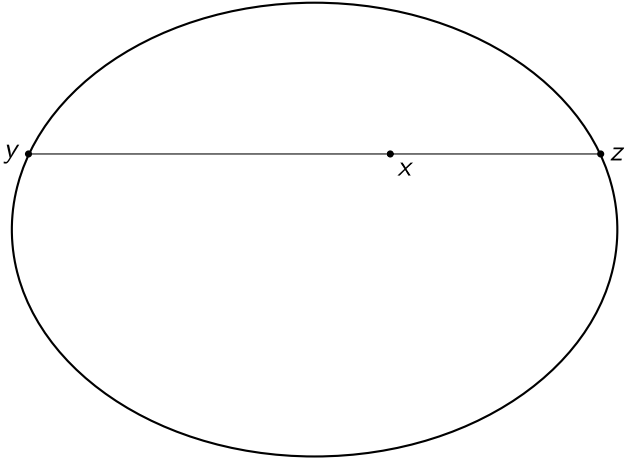

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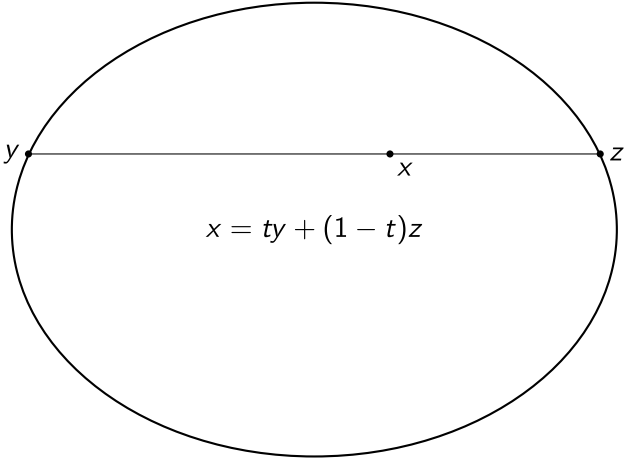

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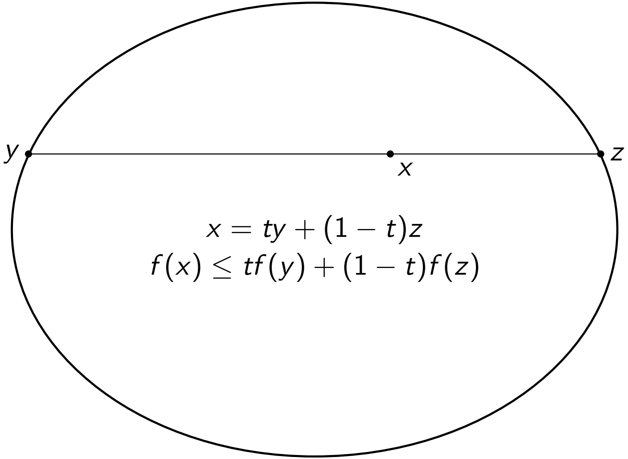

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$$f(x) \leq tf(y) + (1-t)f(z)$$

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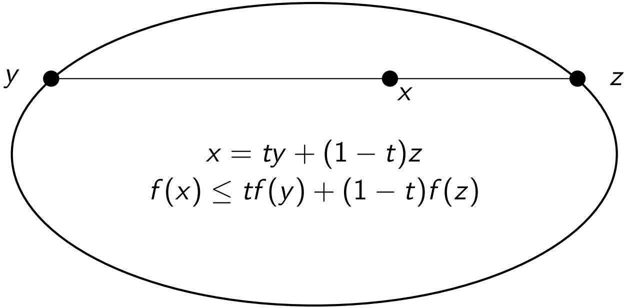

Sending a little bit of Lebesgue mass at x to both y and z.

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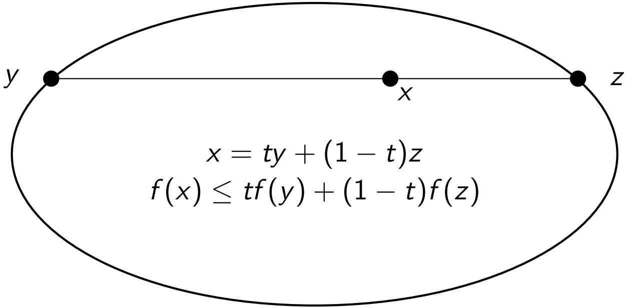

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$$1 \text{ unit of Lebesgue at } x \to \begin{cases} t & \text{at } y \\ 1-t & \text{at } z. \end{cases}$$

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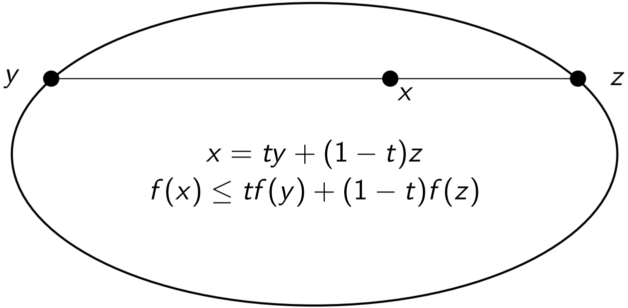

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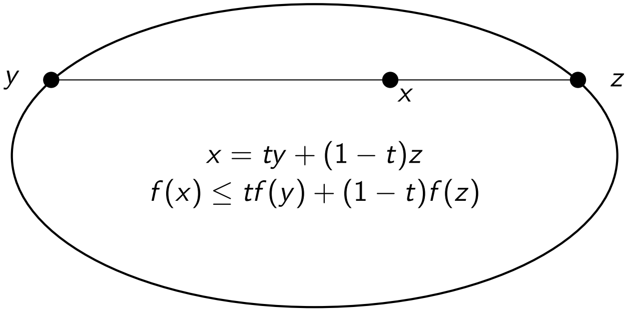
Mechanism to send $\mathcal{H}^d(\Omega)$ to $\mathcal{H}^{d-1}(\partial\Omega)$.

A Transport Problem



Main Question

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Moreover, for all convex $f : \Omega \to \mathbb{R}$, we have

$$\int_\Omega f(x)dx \leq \int_{\partial\Omega} f \; d\nu$$

and in particular, in terms of the Radon-Nikodym derivative,

$$\int_\Omega f \; dx \leq \left\| \frac{d\nu}{d\sigma} \right\|_{L^\infty} \cdot \int_{\partial\Omega} f \; d\sigma$$

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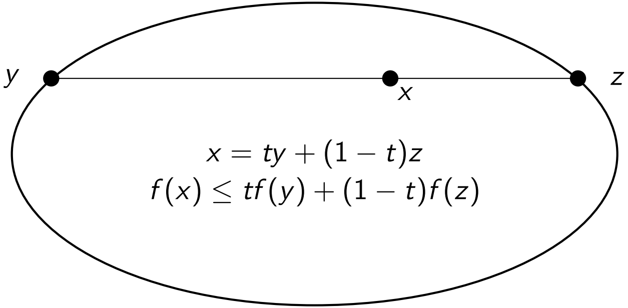

If Ω and $\partial\Omega$ have a different center of mass, then the final measure satisfies

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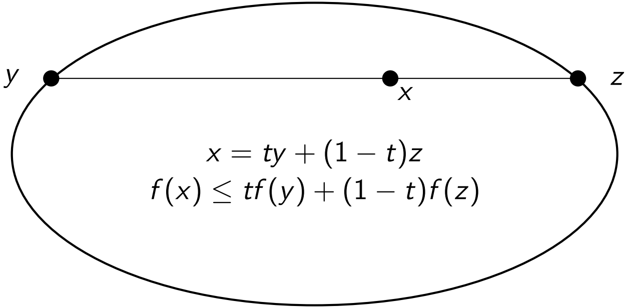

If Ω and $\partial\Omega$ have a different center of mass, then the final measure satisfies

$$\left\| \frac{d\mu}{d\sigma} \right\| > \frac{|\Omega|}{|\partial\Omega|},$$

so it cannot be too evenly distributed. How evenly distributed can it be?

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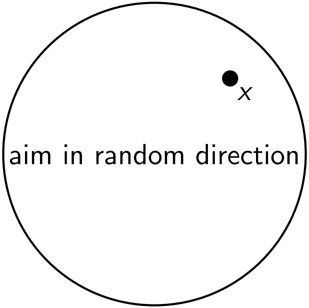

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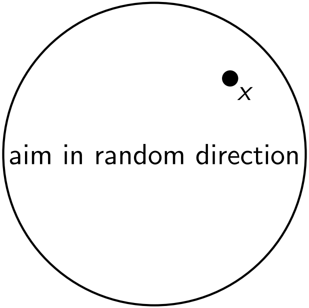

If we distribute the mass randomly in all directions, then the final distribution on the boundary has to be uniform. Thus the constant is 1 for the ball is admissible (and clearly optimal).

Back to subharmonic

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and how does the constant c_{Ω} depend on Ω?

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and how does the constant c_Ω depend on Ω?

We start by recalling some arguments from Niculescu-Persson. To this end, we introduce the function $\phi : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta\phi &= 1 \qquad \text{in } \Omega \\ \phi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Integration by Parts

$$\begin{aligned}\int_\Omega f(x)dx &= \int_\Omega f(x)(-\Delta\phi(x))dx \\ &= \int_\Omega (-\Delta f(x))\phi(x)dx + \int_{\partial\Omega} f(x)\frac{\partial\phi}{\partial n}d\sigma \\ &\le \int_{\partial\Omega} f(x)\frac{\partial\phi}{\partial n}d\sigma\end{aligned}$$

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Find harmonic function corresponding to boundary data given by a characteristic function in the neighborhood where gradient is large.

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$$\int_{\Omega} f(x)dx \leq c_{\Omega} \int_{\partial\Omega} f(x)dx.$$

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$$\int_\Omega f(x)dx \leq c_\Omega \int_{\partial\Omega} f(x)dx.$$

We have

$$c_\Omega \leq \left\| \frac{\partial \phi}{\partial n} \right\|_{L^\infty}$$

where $\phi : \Omega \to \mathbb{R}$ is such that

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So it boils down to understanding the maximal gradient of

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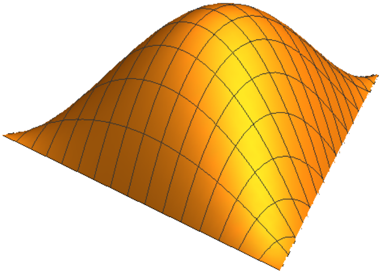

(solution on an equilateral triangle)

This turns out to be a classical problem and there are lots of estimates that are known.

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Theorem (Beck, Brandolini, Burdzy, Henrot, Langford, Larson, Smits, S, 2019)

Let $f : \Omega \to \mathbb{R}$ be positive, $\Delta f \geq 0$ and let Ω be convex. Then

$$\frac{1}{|\Omega|} \int_{\Omega} f dx \leq \frac{c_n}{|\partial\Omega|} \int_{\partial\Omega} f d\sigma,$$

where $n \lesssim c_n \lesssim n^{3/2}$.

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Theorem (Simon Larson, 2020)

Let $f : \Omega \to \mathbb{R}$ be positive, $\Delta f \geq 0$ and let Ω be convex. Then

$$\frac{1}{|\Omega|} \int_\Omega f dx < \frac{n}{|\partial\Omega|} \int_{\partial\Omega} f d\sigma$$

and n is the sharp constant. (No extremizers!)

$$\frac{1}{|\Omega|}\int_\Omega f dx \leq \frac{n}{|\partial\Omega|}\int_{\partial\Omega} f d\sigma$$

is the sharp result in terms of $|\Omega|$ and $|\partial\Omega|$. However, one could also invoke other (or fewer/other) geometric quantities.

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Open Problem: the sharp constant and whether there is an extremal domain is less clear.

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The optimal constant has to satisfy $1/(2\sqrt{\pi e}) \leq c_n \leq 1$. (Lower bound given by ellipsoids, example by Thomas Beck.) It's not entirely clear how extremal domain has to look.

Focusing on $n = 2$

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As it turns out, this question is 165 years old!

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> *Les points dangereux sont donc, comme dans l'ellipse et le rectangle, les points du contour les plus rapproches de l'axe de torsion, ou les extremites des petits diametre. (Saint Venant, 1856)*

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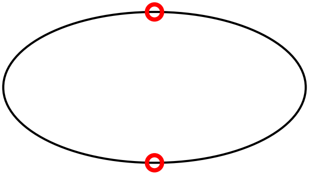

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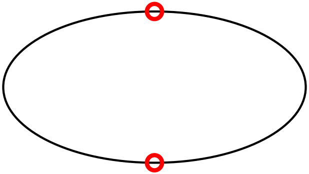

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> *M. de St. Venant also calls attention to a conclusion from his solutions which to many may be startling, that in the simpler cases the places of greatest distortion are those points of the boundary which are nearest to the axis [...] and the places of least distortion those farthest from it.* *(Thomson & Tait, Treatise on Natural Philosophy, 1867)*

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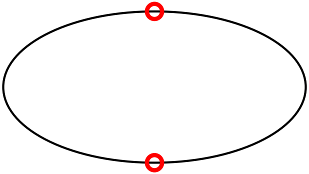

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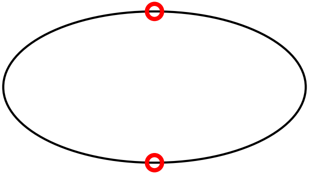

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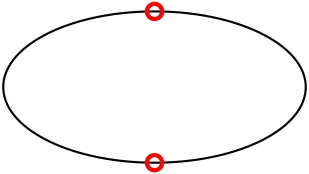

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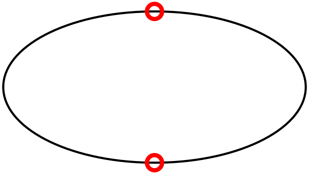

- 1871: Boussinesq gives a heuristic explanation
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- 1920: Griffith & Sir G. I. Taylor build a *soap bubble machine* to compute torsion
- 1930: Polya proves the maximum is on the boundary.

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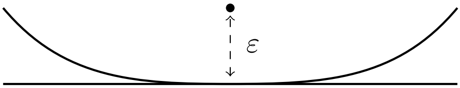

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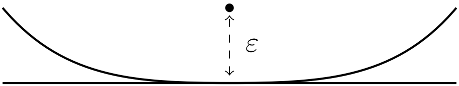

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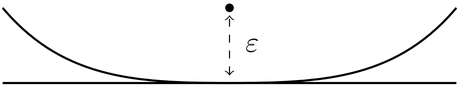

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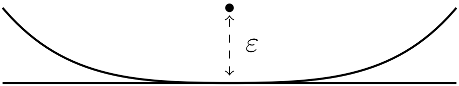

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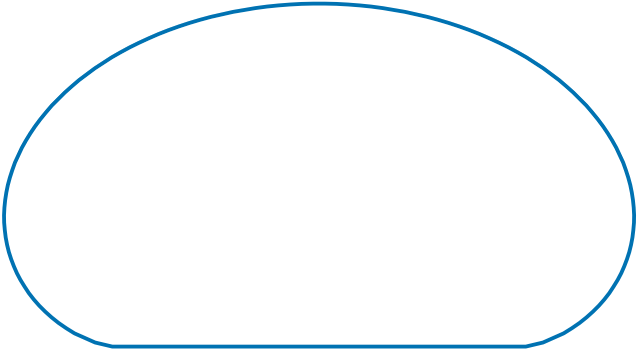
Focusing on $n = 2$

Here's the result of some high precision numerics.



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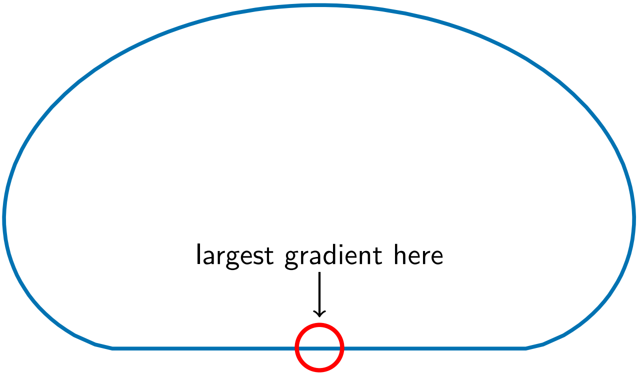

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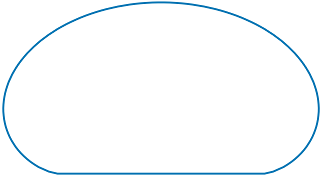

Independent argument proposed by Guido Sweers.

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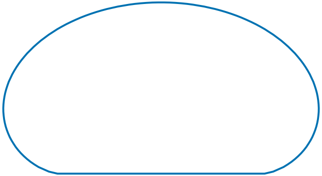

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coming with an explicit construction. Let Ω be a simply connected domain and let $h : E \to \Omega$ be a biconformal map. Then the solutions of

$$\begin{cases} -\Delta w = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta u = |h'(\cdot)|^2 (f \circ h) & \text{in } E \\ u = 0 & \text{on } \partial E \end{cases}$$

are related via

$$(w \circ h)(x, y) = u(x, y).$$

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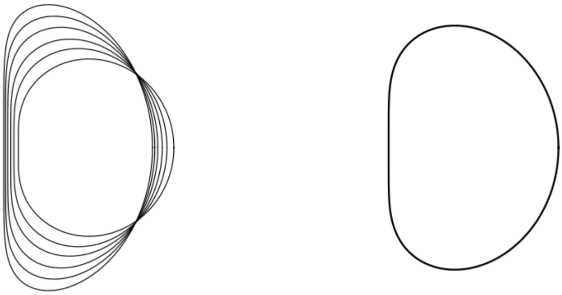
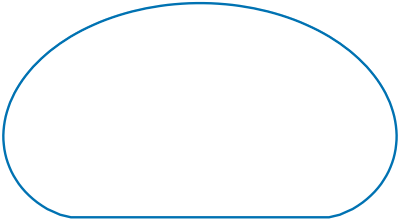


FIGURE 13. Left: $h_q(E_q)$ for $1 \leq q \leq 2$. Right: $h_q(E_q)$ for $q = 1.386$.

Shape Optimization?



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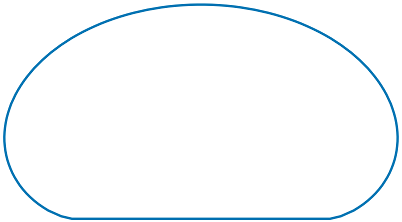

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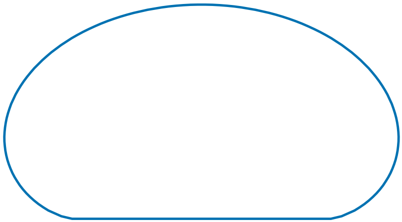

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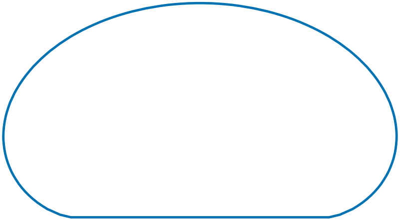

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And what about higher dimensions?

Thank you!