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AGA

RADEMACHER - GAUSSIAN TAIL COMPARISON

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$\varepsilon_1, \varepsilon_2, \dots$ indep. Rademachers $\mathbb{P}(\varepsilon_j = \pm 1) = 1/2$

$g_1, g_2, \dots$ indep. Gaussians $N(0,1)$

$$R = \sum_{j=1}^{n} v_j \varepsilon_j \quad , \quad G = \sum_{j=1}^{n} v_j g_j \sim N(0, \textstyle\sum v_j^2) \quad , \quad v_1, \dots, v_n \in \mathbb{R}$$

- $\mathbb{P}(|R| > t) \leq 2 \exp\left\{ - \dfrac{t^2}{2 \sum a_j^2} \right\}$ Hoeffding

Efron: "not sharp enough to be useful in practice"

$$(*) \quad \mathbb{P}(|R| > t) \leq C\, \mathbb{P}(|G| > t) \qquad \forall t > 0$$

Pinelis '93 $\qquad C = \dfrac{2e^3}{9} \qquad$ (... Eaton, Efron ~'70)

Bobkov - Götze - Houdré '01 $\qquad$ different proof

Bentkus - Dzindzalieta '15 $\qquad$ optimal C $\qquad (n=2,\ a_1 = a_2 = 1,\ t = 2)$

Pinelis '12 $\qquad C = 1 + O(\frac{1}{t})$

$(*)$ holds $\quad \forall n \geq 1 \quad \forall v_1, \dots, v_n \in \mathbb{R}^d \qquad (|\cdot|$ Euclidean norm$)$

s.t. their Gram matrix $A = [\langle v_i, v_j \rangle]_{i,j \leq n}$

has e-val^s $\in \{0, 1\}$

(A is an orthogonal projection)

THM $\quad (*)$ also holds if $\dim \operatorname{span}\{v_1, \dots, v_n\} \leq 2$

$(\operatorname{rank} A \leq 2)$

$\left(C = \dfrac{3e^2}{4}, \ \underset{\text{today}}{4000} \right)$

Conj. $\exists\, C>0$ s.t. $(*)$ holds $\forall n,d\geq 1$ $\forall\, v_1,\dots,v_n\in\mathbb{R}^d$

Any norm? E.g. $\|x\|_1=\sum|x_j|$, $n=d$,

$$\|R\|_1=\Big\|\sum_{j=1}^{d}\varepsilon_j e_j\Big\|_1=d\,,$$

$$\|G\|_1=\Big\|\sum g_j e_j\Big\|_1=\sum_{j=1}^{d}|g_j| \quad\leftarrow \text{concentrated at } \sqrt{\tfrac{2}{\pi}}\cdot d$$

$$\mathbb{P}\left(\|R\|_1\geq d\right)=1\,,\quad \mathbb{P}\left(\|G\|_1\geq d\right)\leq e^{-cd}$$

Oleszkiewicz '14 $\Longrightarrow$ $\mathbb{P}\left(\|R\|>t\right)\leq C\,\mathbb{P}\left(\|G\|>\frac{t}{C}\right)$

strong/weak tails + scalar case

in every sep. Banach space

GOAL: $\mathbb{P}\left(|R|>t\right)\leq C\,\mathbb{P}\left(|G|>t\right)$

Step 0 By Payley-Zygmund type bounds $\left(\mathbb{P}(Y\geq 0)\geq\frac{1}{3}\frac{(\mathbb{E}Y^2)^2}{\mathbb{E}Y^4}\right)$

OK for $t\leq\sqrt{\mathbb{E}|G|^2}$ $\quad\left(Y=|G|^2-\mathbb{E}|G|^2\right)$

Pinelis' approach

Step 1 Markov's ineq.

$$\mathbb{P}\left(|R|>t\right)\leq\frac{1}{f(t)}\,\mathbb{E}f(|R|)\underset{\text{Step 2}}{\leq}\frac{1}{f(t)}\,\mathbb{E}f(|G|)$$

Step 2 $R\rightsquigarrow G$

$$C^2_{\text{conv}}=\left\{f:\mathbb{R}\to\mathbb{R}\ \text{ even, convex with } f''\text{ convex}\right\}$$

sym. $\mathbb{E}g_j^2=1$

$$(**)\qquad \mathbb{E}f\Big(\Big|\sum\varepsilon_j v_j\Big|\Big)\leq\mathbb{E}f\Big(\Big|\sum g_j v_j\Big|\Big)$$

$\forall\, v_1,v_2,\dots\in\mathbb{R}^d$ $\forall f\in C^2_{\text{conv}}$

$$\text{OTT}: \quad \underbrace{\mathbb{E} f(\varepsilon_1)}_{f(1)} \le \underbrace{\mathbb{E} f(g_1)}_{\mathbb{E} f(\sqrt{g_1^2})} \qquad \text{(Jensen)}$$

induction: $|R|^2 = |\sum \varepsilon_j v_j|^2 = \langle \sum \varepsilon_j v_j, \sum \varepsilon_j v_j \rangle = \sum A_{ij} \varepsilon_i \varepsilon_j$

$$= (\alpha \varepsilon_1 + \beta)^2 + b$$

$$\mathbb{E} f(|R|) = \mathbb{E} \frac{1}{2} \underbrace{\left[f\left(\sqrt{(\alpha\varepsilon_1+\beta)^2 + b}\right) + f\left(\sqrt{(\alpha\varepsilon_1 - \beta)^2 + b}\right) \right]}_{\text{as a fun of } \varepsilon_1 \quad \in C^2_{\text{conv}}}$$

$$\le \quad \varepsilon_1 \rightsquigarrow g_1 \quad \le \dots$$

<u>Step 3</u> $\exists f \in C^2_{\text{conv}}$ $\quad \frac{1}{f(t)} \mathbb{E} f(|G|) \overset{(***)}{\le} C\, \mathbb{P}(|G| > t)$

$$|G|^2 = \sum A_{ij} g_i g_j \underset{\text{rot. inv.}}{\overset{d}{=}} \sum \underset{\substack{\uparrow \\ \text{e-val}^s \text{ of } A}}{\lambda_j} g_j^2$$

- $\lambda_j \in \{0,1\}$ $\qquad |G|^2 = \underbrace{g_1^2 + \dots + g_k^2}_{\chi^2(k)}$, $k = \text{rank}(A)$

  Pinelis: $\quad f(t) = (|t| - u)_+^3$ is "optimal" and (***) holds for some $u = u(t,k)$. with $C = \frac{2e^3}{9}$.

- $\text{rank}(A) = 2$, $\qquad |G|^2 \overset{d}{=} \lambda_1 g_1^2 + \lambda_2 g_2^2$

by homog. w/o log. $\lambda_1 = 1$, $\lambda_2 \le 1$

Let $0<\alpha<1$, $X = \sqrt{g_1^2 + \alpha g_2^2}$

LM $\forall t>1 \quad \exists\, 0<u<t \qquad \dfrac{\mathbb{E}(X-u)_+^3}{(t-u)^3} \le 4000 \cdot \mathbb{P}(X>t)$

p = density of X (Bessel fun. I_0)

(1) p is log-concave on (t_0, ∞), $t_0 = 3/4$ (very technical bounds on I_1/I_0 Nåsell '78)

(2) $p(1) > c_0$

Proof of LM: $h(x) = \mathbb{P}(X>x)$ also log-concave on (t_0,∞) (P–L + (1))

$$\mathbb{E}(X-u)_+^3 = \int_u^\infty 3(x-u)^2 h(x)\, dx$$

need this $\forall x>u$ will choose $u>t_0$ $\longrightarrow$ $h(x) \le h(t)\, e^{-a(x-t)}$, $\quad a = (-\log h)'(t)$

$$\le 3h(t) \int_u^\infty (x-u)^2 e^{-a(x-t)}\, dx$$

$$= 6\,h(t) \cdot \underbrace{\frac{e^{a(t-u)}}{a^3}}_{\frac{e^c}{c^3}(t-u)^3}$$

Let $u = t - \frac{c}{a}$ $\quad \left(a = \frac{c}{t-u}\right)$

so $$\frac{\mathbb{E}(X-u)_+^3}{(t-u)^3} \le \underbrace{\frac{6e^c}{c^3}}_{4000} \underbrace{h(t)}_{\mathbb{P}(X>t)}$$

(val. of $\underbrace{c}_{(1-t_0)c_0}$ chosen to make sure $u > t_0$ using (2)) . $\square$