A new connection between the volume product and regularization of heat flow

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This talk is based on a joint work with Shohei Nakamura (Osaka University).
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Overview

- Blaschke–Santaló inequality and Mahler conjecture for volume products: $\sup_K |K||K^\circ|$, $\inf_K |K||K^\circ|$.

- Smoothing property: For $s > 0$ (time) and $f\colon \mathbb{R}^n \to [0,\infty)$ (initial data),

$$P_s f(x) := \int_{\mathbb{R}^n} f(e^{-s}x + \sqrt{1-e^{-2s}}y)\, d\gamma(y),$$

  where $d\gamma(y) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}|y|^2}\, dy$. Then $\partial_s u = \Delta u - \langle x, \nabla u\rangle$ with $u := P_s f$.

$$\|P_s f\|_{L^q(\gamma)} \leq \text{ or } \geq \|f\|_{L^p(\gamma)}$$

  for $q, p \in \mathbb{R}$.

- Our observation: A specific smoothing property yields Blaschke–Santaló inequality / Mahler conjecture.

- Volume products are embedded into the Brascamp–Lieb inequality, which leads to the Kolesnikov–Werner conjecture (Blaschke–Santaló type inequality for many convex bodies).
  $\rightsquigarrow$ Shohei Nakamura (Osaka) will give a talk bout this topic at Online AGA seminar on November 16.

Volume product

- Let $K \subset \mathbb{R}^n$ be a convex body (i.e., compact and convex set with $\mathrm{int} K \neq \emptyset$) with $0 \in \mathrm{int} K$.
- Polar body of K:

$$K^\circ := \{x \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1, \forall y \in K\}.$$

c.f. $(\mathrm{B}_p^n)^\circ = \mathrm{B}_{p'}^n$ with $p^{-1} + (p')^{-1} = 1$ where

$$\mathrm{B}_p^n := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} \leq 1\}, \quad 1 \leq p \leq \infty.$$

- Volume product of K:

$$v(K) := |K||K^\circ|.$$

  v is linear invariant, i.e., $v(TK) = v(K)$ for any linear isomorphism T on $\mathbb{R}^n$.

Blaschke–Santaló inequality and Mahler conjecture

Theorem 1 (Blaschke 1917, Santaló 1949, Petty 1985)

For any convex body $K \subset \mathbb{R}^n$ *with* $b_K := \frac{1}{|K|}\int_K x\,dx = 0$, *it holds that*

$$v(K) \leq v(\mathrm{B}_2^n).$$

Equality holds iff K *is a symmetric ellipsoid.*

Mahler conjecture

- *Non-symmetric case : For any convex body* $K \subset \mathbb{R}^n$ *with* $b_K = 0$,

  $$v(K) \geq v(\Delta_0^n),$$

  where Δ_0^n *is an* n*-dimensional simplex with* $b_{\Delta_0^n} = 0$.
- *Symmetric case : For any symmetric convex body* $K \subset \mathbb{R}^n$ *(i.e.,* $K = -K$*),*

  $$v(K) \geq v(\mathrm{B}_\infty^n) = v(\mathrm{B}_1^n).$$

Known results

- Mahler (1938): symmetric and non-symmetric cases for $n = 2$.
- Iriyeh–Shibata (2020): symmetric case for $n = 3$. A short proof by Fradelizi–Hubard–Meyer–Roldán-Pensado–Zvavitch (2022).

Partial answers.

- unconditional convex bodies: Saint-Raymond (1980), Meyer (1986).
- zonoid: Reisner (1986), Gordon–Meyer–Reisner (1988).
- symmetric polytopes in $\mathbb{R}^n$ with $2n+2$ vertices: Lopez and Reisner (1998), Karasev (2021).
- polytopes with not more than $n+3$ vertices in $\mathbb{R}^n$: Meyer–Reisner (2006).
- some bodies with many symmetries: Barthe–Fradelizi (2013), Iriyeh–Shibata (2022).
- Asymptotic estimate: Bourgain–Milman (1986), Kuperberg (2008).

Functional BS and Mahler conjecture

- Functional volume product: For $e^{-\psi} \in L^1(dx)$,

$$\int_{\mathbb{R}^n} e^{-\psi}\, dx \int_{\mathbb{R}^n} e^{-\psi^*}\, dx,$$

where ψ^* is the Legendre transform of ψ, i.e.,

$$\psi^*(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y\rangle - \psi(y)].$$

- Set $\|x\|_K := \inf\{\lambda > 0 \mid x \in \lambda K\}$ for a convex body $K \subset \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\|x\|_K^2}\, dx = 2^{\frac{n}{2}}\Gamma(\frac{n}{2}+1)|K|$$

and

$$(\frac{1}{2}\|x\|_K^2)^* = \frac{1}{2}\|x\|_{K^\circ}^2.$$

$\rightsquigarrow$

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\|x\|_K^2}\, dx \int_{\mathbb{R}^n} e^{-\frac{1}{2}\|x\|_{K^\circ}^2}\, dx = 2^n\Gamma(\frac{n}{2}+1)^2 v(K).$$

Functional BS and Mahler conjecture

Theorem 2 (Ball 1986, Artstein-Avidan–Klartag–Milman 2004, Lehec 2009)

For any $e^{-\psi} \in L^1(dx)$ *with* $\int_{\mathbb{R}^n} x e^{-\psi}\, dx = 0$,

$$\int_{\mathbb{R}^n} e^{-\psi}\, dx \int_{\mathbb{R}^n} e^{-\psi^*}\, dx \leq (2\pi)^n.$$

Equality holds iff $\psi(x) = \langle Ax, x \rangle + a$ *for some* $A \in \mathrm{Sym}^+(\mathbb{R}^n)$ *and* $a \in \mathbb{R}$.

Functional Mahler conjecture (Fradelizi–Meyer 2008)

- *Non-symmetric case: For any convex function* $\psi \colon \mathbb{R}^n \to \mathbb{R}$ *with* $0 < \int_{\mathbb{R}^n} e^{-\psi}\, dx < +\infty$,

$$\int_{\mathbb{R}^n} e^{-\psi}\, dx \int_{\mathbb{R}^n} e^{-\psi^*}\, dx \geq e^n.$$

- *Symmetric case: For any symmetric convex function* $\psi \colon \mathbb{R}^n \to \mathbb{R}$ *with* $0 < \int_{\mathbb{R}^n} e^{-\psi}\, dx < +\infty$,

$$\int_{\mathbb{R}^n} e^{-\psi}\, dx \int_{\mathbb{R}^n} e^{-\psi^*}\, dx \geq 4^n.$$

Known results

- Fradelizi–Meyer (2008, 2010): symmetric and non-symmetric cases for $n = 1$.
- Fradelizi–Nakhle (2022): symmetric case for $n = 2$.

<u>Partial answers.</u>

- Unconditional convex function: Fradelizi–Meyer (2008).
- Asymptotic estimate: Klartag–Milman (2005), Fradelizi–Meyer (2008).

Hypercontractivity

- For $s > 0$ (time) and $f\colon \mathbb{R}^n \to [0,\infty)$ (initial data), the Ornstein–Uhlenbeck semigroup is given by

$$P_s f(x) := \int_{\mathbb{R}^n} f(e^{-s}x + \sqrt{1-e^{-2s}}y)\, d\gamma(y),$$

  which is a solution of $\partial_s u = \Delta u - \langle x, \nabla u\rangle$ with $u(s,x) = P_s f(x)$.
- Mass-preservation: $\|P_s f\|_{L^1(\gamma)} = \|f\|_{L^1(\gamma)}$ and $\lim_{s\to\infty} P_s f \equiv \text{const}$.
- Contraction: $\|P_s f\|_{L^p(\gamma)} \le \|f\|_{L^p(\gamma)}$ for $p \ge 1$ and $\|P_s f\|_{L^p(\gamma)} \ge \|f\|_{L^p(\gamma)}$ for $-\infty < p \le 1$. In particular,

$$\begin{aligned} 1 \le q \le p \quad &\Rightarrow \quad \|P_s f\|_{L^q(\gamma)} \le \|P_s f\|_{L^p(\gamma)} \le \|f\|_{L^p(\gamma)}, \\ -\infty < p \le q \le 1 \quad &\Rightarrow \quad \|P_s f\|_{L^q(\gamma)} \ge \|P_s f\|_{L^p(\gamma)} \ge \|f\|_{L^p(\gamma)}. \end{aligned}$$

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Question. Can we exceed $1 \leq q \leq p$ or $-\infty \leq p \leq q \leq 1$?

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$\rightsquigarrow$ **Yes!** This is Hypercontractiviy.

Hypercontractivity

Theorem 3 (Nelson's forward / Borell's reverse HC)

Suppose $s > 0$ *and* $p, q \in \mathbb{R} \setminus \{0\}$. *Then*

$$1 < p, q \text{ with } \frac{q-1}{p-1} \leq e^{2s} \quad \Rightarrow \quad \|P_s f\|_{L^q(\gamma)} \leq \|f\|_{L^p(\gamma)}.$$

Moreover

$$1 < p, q \text{ with } \frac{q-1}{p-1} > e^{2s} \quad \Rightarrow \quad \sup_{0 \leq f \in L^p(\gamma)} \frac{\|P_s f\|_{L^q(\gamma)}}{\|f\|_{L^p(\gamma)}} = +\infty.$$

Similarly,

$$-\infty < p, q < 1 \text{ with } \frac{q-1}{p-1} \leq e^{2s} \quad \Rightarrow \quad \|P_s f\|_{L^q(\gamma)} \geq \|f\|_{L^p(\gamma)}.$$

Moreover

$$-\infty < p, q < 1 \text{ with } \frac{q-1}{p-1} > e^{2s} \quad \Rightarrow \quad \inf_{0 \leq f \in L^p(\gamma)} \frac{\|P_s f\|_{L^q(\gamma)}}{\|f\|_{L^p(\gamma)}} = 0.$$

- p, q satisfy Nelson's time if $\frac{q-1}{p-1} \leq e^{2s}$.
- In below, we reformulate HC as $\|P_s[f^{\frac{1}{p}}]\|_{L^q(\gamma)} \leq$ or $\geq (\int_{\mathbb{R}^n} f \, d\gamma)^{\frac{1}{p}}$.

HC with improved Nelson's time ⇒ volume product

Proposition 1 (Nakamura–T.)

Suppose that for small $s > 0$, there exists some $q_s = -2s + o(s) < 0$, $p_s = 2s + o(s) > 0$ and $C_{\mathrm{BS}}(s) > 0$ such that

$$\|P_s[f^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)} \geq C_{\mathrm{BS}}(s)^{\frac{1}{p_s}} \left(\int_{\mathbb{R}^n} f\, d\gamma\right)^{\frac{1}{p_s}}$$

for all nonnegative symmetric function f. Then

$$\int_{\mathbb{R}^n} e^{-\psi}\, dx \int_{\mathbb{R}^n} e^{-\psi^*}\, dx \leq (2\pi)^n \liminf_{s\downarrow 0} (C_{\mathrm{BS}}(s)^{-1})$$

for all symmetric function ψ.
Similarly, suppose that for small $s > 0$, there exists some $q_s = -2s + o(s) < 0$, $p_s = 2s + o(s) > 0$ and $C_{\mathrm{IS}}(s) > 0$ such that

$$\|P_s[f^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)} \leq C_{\mathrm{IS}}(s)^{\frac{1}{p_s}} \left(\int_{\mathbb{R}^n} f\, d\gamma\right)^{\frac{1}{p_s}}$$

for all nonnegative log-concave function f. Then

$$\int_{\mathbb{R}^n} e^{-\psi}\, dx \int_{\mathbb{R}^n} e^{-\psi^*}\, dx \geq (2\pi)^n \limsup_{s\downarrow 0} (C_{\mathrm{IS}}(s)^{-1})$$

for all convex function ψ.

HC with improved Nelson's time $\Rightarrow$ volume product

- This link is motivated by Bobkov–Gentil–Ledoux's argument (2001): HC for heat flow yields HC for Hamilton–Jacobi semigroups as $s \downarrow 0$.
- If $q_s = -2s + o(s) < 0$ and $p_s = 2s + o(s) > 0$, then

$$\frac{q_s - 1}{p_s - 1} = 1 + 4s + o(s) > e^{2s}.$$

Thus p_s, q_s do NOT satisfy Nelson's time! (Recall that we cannot exceed Nelson's time in general.)

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Question 1. Can we exceed Nelson's time for specific functions f?

$$-\infty < q < p < 1 \text{ with } \frac{q-1}{p-1} > e^{2s} \quad \Rightarrow \quad \|P_s[f^{\frac{1}{p}}]\|_{L^q(\gamma)} \geq \left(\int_{\mathbb{R}^n} f \, d\gamma\right)^{\frac{1}{p}}.$$

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$\rightsquigarrow$ **Yes!** if f is symmetric (or more generally, its barycenter is 0).

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Question 2. Does forward HC hold for $-\infty < p, q < 1$?.

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$\rightsquigarrow$ **Yes!** if f has a strong log-concavity and convexity under Nelson's time.

Answer to Question 1: Improved reverse HC

Theorem 4 (Nakamura–T.)

Let $s > 0, p, q < 1$ satisfy $0 < p \leq 1 - e^{-2s}$ and $-p \leq q < 0$. Then for any nonnegative function $f \in L^1(\gamma)$ with $\int_{\mathbb{R}^n} xf\, d\gamma = 0$, it holds that

$$\|P_s[f^{\frac{1}{p}}]\|_{L^q(\gamma)} \geq \Big(\int_{\mathbb{R}^n} f\, d\gamma\Big)^{\frac{1}{p}}.$$

- For example,

$$p = 1 - e^{-2s}, \;\; q = -p \quad \Rightarrow \quad \frac{q-1}{p-1} > e^{2s}.$$

- The proof is accomplished by combing the Prékopa–Leindler ineq., Wang's Harnack ineq. and Yao–Yao partition (Lehec's approach to BS).
- Furthermore, we can recently exceed the time relation above (in preparation):

$$0 < p \leq 1 - e^{-2s}, 1 - e^{2s} \leq q < 0 \quad \Rightarrow \quad \inf_{\substack{0 \leq f \in L^1(\gamma),\\ \text{sym. log-concave}}} \frac{\|P_s[f^{\frac{1}{p}}]\|_{L^q(\gamma)}}{(\int_{\mathbb{R}^n} f\, d\gamma)^{\frac{1}{p}}} = 1.$$

  This relation is optimal. The proof is accomplished by the flow monotonicity of the Fokker–Planck flow.

Sketch of the proof for Question 1

Suppose that $p = 1 - e^{-2s}$, $q = -p$ and that f is unconditional, i.e., $f(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) = f(x_1, \ldots, x_n)$ for $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$. Then it suffices to show

$$\left(2^n \int_{\mathbb{R}^n_+} P_s f d\gamma\right)^{\frac{1}{2}} \left(2^n \int_{\mathbb{R}^n_+} P_s [f^{1/p}]^{-p} d\gamma\right)^{\frac{1}{2}} \leq 1.$$

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STEP 1. Use the multiplicative Prékopa–Leindler inequality, i.e., given Borel measurable functions $F, G : \mathbb{R}^n_+ \to \mathbb{R}_+$,

$$\left(\int_{\mathbb{R}^n_+} F\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n_+} G\, dx\right)^{\frac{1}{2}} \leq \int_{\mathbb{R}^n_+} \sup_{\substack{z_i = \sqrt{x_i y_i} \\ x, y \in \mathbb{R}^n_+}} F(x)^{\frac{1}{2}} G(y)^{\frac{1}{2}}\, dz.$$

Using this,

$$\left(\int_{\mathbb{R}^n_+} P_s f d\gamma\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n_+} P_s[f^{1/p}]^{-p} d\gamma\right)^{\frac{1}{2}}$$
$$\leq \int_{\mathbb{R}^n_+} \sup_{\substack{z_i = \sqrt{x_i y_i} \\ x, y \in \mathbb{R}^n_+}} (P_s f(x)\gamma(x))^{\frac{1}{2}} (P_s[f^{1/p}]^{-p}(y)\gamma(y))^{\frac{1}{2}}\, dz.$$

Sketch of the proof for Question 1

STEP 2. Use Wang's Harnack inequality, i.e., given $\alpha > 1$ and $h\colon \mathbb{R}^n \to \mathbb{R}_+$,

$$(P_s h(x))^\alpha \leq P_s[h^\alpha](y) \exp\left(\frac{\alpha|x-y|^2}{2(\alpha-1)(e^{2s}-1)}\right), \quad \forall x, y \in \mathbb{R}^n.$$

Using this as $\alpha = \frac{1}{p} > 1$,

$$P_s f(x)^{\frac{1}{p}} \leq P_s[f^{1/p}](y) \exp\left(\frac{\frac{1}{p}|x-y|^2}{2(\frac{1}{p}-1)(e^{2s}-1)}\right) = P_s[f^{1/p}](y) e^{\frac{1}{2p}|x-y|^2}.$$

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STEP 3. Combining above,

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n_+} P_s f d\gamma\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n_+} P_s[f^{1/p}]^{-p} d\gamma\right)^{\frac{1}{2}} \\
&\le \int_{\mathbb{R}^n_+} \sup_{\substack{z_i = \sqrt{x_i y_i} \\ x, y \in \mathbb{R}^n_+}} e^{\frac{1}{4}|x-y|^2} \gamma(x)^{\frac{1}{2}} \gamma(y)^{\frac{1}{2}}\, dz \\
&= \int_{\mathbb{R}^n_+} d\gamma(z) = \frac{1}{2^n}.
\end{aligned}$$

Answer to Question 2: Forward HC for $p, q < 1$

Theorem 5 (Nakamura–T.)

Let $s > 0$, $0 < p < 1$, $q \in (-\infty, 1) \setminus \{0\}$ *satisfy* $\frac{q-1}{p-1} = e^{2s}$, *and* $\beta \geq 1$. *Then for any* $f\colon \mathbb{R}^n \to (0, \infty)$ *satisfying*

$$0 \leq \nabla^2 \log f \leq (1 - \frac{1}{\beta})\mathrm{id}_{\mathbb{R}^n},$$

it holds that

$$\|P_t[f^{\frac{1}{p}}]\|_{L^q(\gamma)} \leq \|P_t[(\frac{\gamma_\beta}{\gamma})^{\frac{1}{p}}]\|_{L^q(\gamma)} (\int_{\mathbb{R}^n} f \, d\gamma)^{\frac{1}{p}}.$$

Here

$$\gamma_\beta(x) := \frac{1}{(2\pi\beta)^{\frac{n}{2}}} e^{-\frac{1}{2\beta}|x|^2}.$$

- The proof is accomplished by the flow monotonicity of the Fokker–Planck flow combined with the Poincaré inequality.

New lower bound for specific volume products

Corollary 1 (Nakamura–T.)

Let $n \geq 2$, $\kappa \in (0,1]$ and $K \subset \mathbb{R}^n$ be a convex body with $0 \in \mathrm{int} K$. Suppose that $\|\cdot\|_K^2$ is C^2 on $\mathbb{R}^n \setminus \{0\}$ and satisfies

$$\nabla^2(\frac{1}{2}\|\cdot\|_K^2) \geq \kappa\Lambda^{-1}, \quad \nabla^2(\frac{1}{2}\|\cdot\|_{K^\circ}^2) \geq \kappa\Lambda$$

for some positive definite symmetric matrix $\Lambda \in \mathbb{R}^{n \times n}$. Then it holds that

$$v(K) \geq (\kappa^2 e^{1-\kappa^2})^{\frac{n}{2}} v(\mathrm{B}_2^n).$$

- Our assumptions imply that the principle curvatures on $\partial(\Lambda^{-\frac{1}{2}}K)$ and $\partial(\Lambda^{-\frac{1}{2}}K)^\circ$ are uniformly bounded from below by κ.
- Stancu (2009) and Reisner–Schütt–Werner (2012): The boundary of the local minimizer must be flat, i.e., if there exists a point in either ∂K or ∂K° at which the (generalized) Gauss curvature exists and is not 0 then $v(K)$ is not a local minimum.
- Mahler's conjecture is true for K satisfying our assumptions with κ close to 1, i.e.,

$$(\kappa^2 e^{1-\kappa^2})^{\frac{n}{2}} v(\mathrm{B}_2^n) \geq v(\Delta_0^n) \quad \text{in non-symmetric case,}$$

$$(\kappa^2 e^{1-\kappa^2})^{\frac{n}{2}} v(\mathrm{B}_2^n) \geq v(\mathrm{B}_\infty^n) \quad \text{in symmetric case.}$$

Sketch of the proof for Question 2

Let v_t be a β-Fokker–Planck solution with $v_0 = f\gamma$, i.e.,

$$\partial_t v_t = \mathcal{L}_\beta^* v_t := \beta \Delta v_t + \mathrm{div}(x v_t).$$

Consider

$$\widetilde{v}_t(x) := P_s[(\frac{v_t}{\gamma})^{\frac{1}{p}}]^q(x)\gamma(x), \quad x \in \mathbb{R}^n.$$

Theorem 6 (Nakamura–T.)

Let $s > 0$, $0 < p < 1 - e^{-2s}$ and $q < 0$ such that $\frac{q-1}{p-1} = e^{2s}$ and $\beta \geq 1$ satisfy

$$\beta_{s,p} := 1 + (\beta - 1)\frac{q}{p}e^{-2s} > 0.$$

If f satisfies $0 \leq \nabla^2 \log f \leq (1 - \frac{1}{\beta})\mathrm{id}_{\mathbb{R}^n}$, then $\partial_t \widetilde{v}_t - \mathcal{L}_{\beta_{s,p}}^* \widetilde{v}_t \leq 0$.

Sketch of proof. Set

$$Q(t) := \int_{\mathbb{R}^n} \widetilde{v}_t \, dx.$$

Then

$$Q'(t) = \int_{\mathbb{R}^n} \partial_t \widetilde{v}_t \, dx \leq \int_{\mathbb{R}^n} \mathcal{L}_{\beta_{s,p}}^* \widetilde{v}_t \, dx = 0.$$

Hence

$$\int_{\mathbb{R}^n} P_s[f^{\frac{1}{p}}]^q \, d\gamma = Q(0) \geq Q(\infty) = \left(\int_{\mathbb{R}^n} P_s[(\frac{\gamma_\beta}{\gamma})^{\frac{1}{p}}]^q \, d\gamma\right)\left(\int_{\mathbb{R}^n} f \, d\gamma\right)^{\frac{q}{p}}.$$

Open question toward Mahler conjecture

Conjecture

Let $s > 0$, $p_s = 1 - e^{-2s}$ *and* $q_s = 1 - e^{2s}$.
Non-symmetric case:

$$\sup_{\substack{0 \le f \in L^1(\gamma) \\ \text{log-concave}}} \frac{\|P_s[f^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)}}{\left(\int_{\mathbb{R}^n} f \, d\gamma\right)^{\frac{1}{p_s}}} = \frac{\|P_s[f_*^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)}}{\left(\int_{\mathbb{R}^n} f_* \, d\gamma\right)^{\frac{1}{p_s}}}$$

where $f_*(x) := \mathbf{1}_{[-1,\infty)^n} e^{-(x_1 + \cdots + x_n)}/\gamma(x)$.
Symmetric case:

$$\sup_{\substack{0 \le f \in L^1(\gamma) \\ \text{sym. log-concave}}} \frac{\|P_s[f^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)}}{\left(\int_{\mathbb{R}^n} f \, d\gamma\right)^{\frac{1}{p_s}}} = \frac{\|P_s[f_{**}^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)}}{\left(\int_{\mathbb{R}^n} f_{**} \, d\gamma\right)^{\frac{1}{p_s}}}$$

where $f_{**}(x) := e^{-(|x_1| + \cdots + |x_n|)}/\gamma(x)$.

- $f_*\gamma, f_{**}\gamma$ are conjectured as minimizers of functional Mahler conjecture.

Thank you for your attention!