The general dual-polar Orlicz-Minkowski problem

Sudan Xing

(Joint with Deping Ye and Baocheng Zhu)

I. Background.

$\mathcal{K}^n_{(o)}$: Convex bodies (compact convex sets) with the origin "o" in their interiors.

S^{n-1} $:= \partial B^n$, the unit sphere.

h_K : $S^{n-1} \to \mathbb{R}$, support function for $K \in \mathcal{K}^n_{(o)}$

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle \text{ for any } u \in S^{n-1}.$$

ρ_K : $S^{n-1} \to \mathbb{R}$, radial function for $K \in \mathcal{K}^n_{(o)}$

$$\rho_K(u) = \max\{\lambda > 0 : \lambda u \in K\} \text{ for any } u \in S^{n-1}$$

$V(\cdot)$: volume (Lebesgue measure)

$$V(K) = \frac{1}{n}\int_{S^{n-1}} h_K(u)\, dS_K(u) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(u)\, du.$$

$S_K(\cdot)$: $S^{n-1} \to \mathbb{R}$, surface area measure for $K \in \mathcal{K}^n_{(o)}$.

K^* $:= \{x \in \mathbb{R}^n : \langle x, y \rangle \le 1 \text{ for all } y \in K\}$

1. $K^* \in \mathcal{K}^n_{(o)}$, polar body
2. $(K^*)^* = K$
3. $\rho_K(u)\, h_{K^*}(u) = 1$, $u \in S^{n-1}$.

II Orlicz-Minkowski problem

Under what conditions on parameters:

μ : a finite Borel measure,

$\phi : (0,\infty) \longrightarrow (0,\infty)$,

does there exist a $K \in \mathcal{K}_o^n$ such that

$$d\mu = \tau \phi(h_K)\, dS_K(\cdot)$$

for some constant τ ?

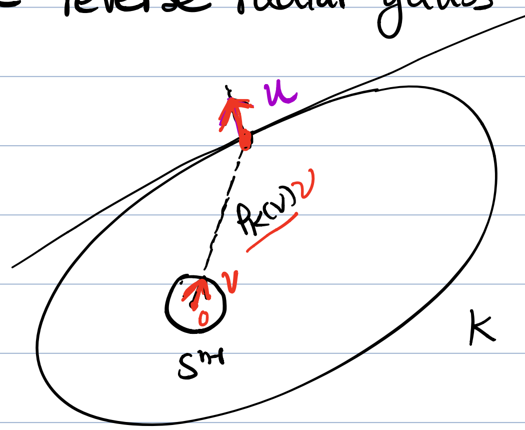
① Haberl-LYZ, ADV, 2010.

② $\phi(t) = t^{1-p}$, the L_p Minkowski problem. Lutwak, JDG, 1993.

③ $\inf \{ \int_{S^{n-1}} \phi(h_K)\, d\mu : \underbrace{V(K)}_{\text{Volume}} = V(B^n) \}$.

III. Dual Orlicz-Minkowski problem.

① The reverse radial gauss image $\alpha_K^*(\cdot)$.



$$u \underset{\alpha_K(\cdot)}{\overset{\alpha_K^*(\cdot)}{\rightleftarrows}} v$$

② The general dual Orlicz curvature measure $\widetilde{C}_{G,\psi}$

$\psi : (0,\infty) \longrightarrow (0,\infty)$ continuous

$G(t,u) : (0,\infty) \times S^{n-1} \longrightarrow (0,\infty)$ continuous

$G_t(t,u) = \frac{\partial G(t,u)}{\partial t}$ integrable on S^{n-1}.

For $K\in\mathcal{K}^n_{(o)}$, $\eta\subseteq S^{n-1}$, the general dual Orlicz curvature measure is

$$\widetilde{C}_{G,\psi}(K,\eta)=\frac{1}{n}\int_{\alpha_K^*(\eta)}\frac{\rho_K(u)\,G_t(\rho_K(u),u)}{\psi(h_K(\alpha_K(u)))}\,du.$$

③ The general dual volume. $\widetilde{V}_G(\cdot)$

$$\widetilde{V}_G(K)=\int_{S^{n-1}}G(\rho_K(u),u)\,du$$

④ The general dual Orlicz-Minkowski problem.

Under what conditions on

μ: a finite Borel measure

$G(t,u)$: $(0,\infty)\times S^{n-1}\to(0,\infty)$

ψ: $(0,\infty)\to(0,\infty)$,

does there exist a $K\in\mathcal{K}^n_{(o)}$ such that

$$\mu=\tau\,\widetilde{C}_{G,\psi}(K,\cdot)$$

for some constant τ?

Remark: a) Gardner-Hug-Weil-Xing-Ye, CVPDE, 2019.
Gardner-Hug-Xing-Ye, CVPDE, 2020.

b) If $G(t,u)=t^{q-1}$, $\psi(t)=t^p$, it recovers the dual L_p Minkowski problem by LYZ, ADV, 2018,

$$\mu=\widetilde{C}_{p,q}(K,\cdot)\ ?$$

c) $\inf\left\{\int_{S^{n-1}}\phi(h_K)\,d\mu:\ \widetilde{V}_G(K)=\widetilde{V}_G(B^n)\right\}$

The general dual volume.

IV. The general dual-polar Orlicz-Minkowski problem.

① Motivation.

Dual L_p Minkowski Prob. $\xleftarrow{\text{dual}}$ L_p Minkowski Prob. $\xrightarrow{\text{Polar}}$ L_p Petty bodies

General dual-polar
Orlicz-Minkowski problem

Dual Orlicz-Minkowski Prob. $\xleftarrow{\text{dual}}$ Orlicz-Minkowski Prob. $\xrightarrow{\text{Polar}}$ Polar Orlicz-Minkowski Prob.

Polar Orlicz-Minkowski problem, Luo-Ye-Zhu, IUMJ, 2020

② Problem: Under what conditions on parameters.

μ: a finite Borel measure

$G: (0,\infty)\times S^{n-1} \to (0,\infty)$

$\varphi: (0,\infty) \to (0,\infty)$

can we find a $K \in \mathcal{K}_{(o)}^n$ solving

$$\inf/\sup\left\{ \int_{S^{n-1}} \varphi(h_K)\,d\mu \;:\; \widetilde{V}_G(K^*) = \widetilde{V}_G(B^n) \right\} ?$$

③ Solution:

μ: a nonzero finite Borel measure not concentrated on any closed hemisphere

φ: consistinuous, increasing and $\lim_{t\to 0^+}\varphi(t) = 0$, $\varphi(1) = 1$, $\lim_{t\to\infty}\varphi(t) = \infty$.

$G(t,u)$: continuous, increasing on t,

$\lim_{t\to 0^+} G(t,\cdot)=0$, $\lim_{t\to\infty} G(t,\cdot)=\infty$ and

$\inf\{G_q(t,u)=G(t,u)/t^q,\ t\geq 1,\ u\in S^{n-1}\}>0$

for some $\underline{q\geq n-1}$.

In this case, there exists a $\boxed{K}\in\mathcal{K}^n_{(o)}$, satisfy

$$\widetilde{V}_G(K^*)=\widetilde{V}_G(B^n)$$

$\underline{\int_{S^{n-1}}\varphi(h_K)d\mu}=\inf\{\int_{S^{n-1}}\varphi(h_Q)\,d\mu : \widetilde{V}_G(Q^*)=\widetilde{V}_G(B^n)\}.$

④ Main point: $\inf \neq -\infty$

$\sup \neq \infty$

$\widetilde{V}_G(K^*)=\widetilde{V}_G(B^n)$.

$\underline{o\in \operatorname{Int} K}$, $K\in\mathcal{K}^n_{(o)}$.

☆

⑤ Main steps:

a) $\mu \xleftarrow{\text{Weakly convergent}} \mu_i$ (discrete measures).

b) (μ_i, G, φ) satisfy the above conditions, there exists Polytopes $P_i\in\mathcal{K}^n_{(o)}$. [$o\in \operatorname{Int} P_i$].

c) (G,φ) make $P_i\subseteq RB^n$ for some $R>0$.

By Blaschke selection theorem, there exists a $K\subseteq\mathbb{R}^n$ and $P_{i_j}\longrightarrow K$, $j\to\infty$.

d) (G,φ) guarantee that $o\in\operatorname{Int}K$ and $K\in\mathcal{K}^n_{(o)}$.

⑥ Further results.

$\mu = S(K,\cdot)$

$\mu = \widetilde{C}_{\varphi,q}(K,\cdot)$

$\mu = \sum_{i=1}^{m} \lambda_i \delta_{u_i}$, where $\{u_1, u_2, \cdots u_m\}$ not concentrated on any closed hemisphere.

φ: continuous, decreasing and
$\lim_{t \to 0^+} \varphi(t) = \infty$, $\varphi(1) = 1$, $\lim_{t \to \infty} \varphi(t) = 0$,

G: satisfy the same condition as above.

then

$$\inf \{ \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) : \widetilde{V}_G(Q^*) = \widetilde{V}_G(B^n) \} = 0 .$$
$$\sup \{ \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) : \widetilde{V}_G(Q^*) = \widetilde{V}_G(B^n) \} = \infty .$$

① Variations of the general dual-polar Orlicz-Minkowski Prob.

a) The general volume $V_G(\cdot)$,
$$V_G(K) = \int_{S^{n-1}} G(h_K(u), u) \, dS_K(u)$$

b) The homogeneous general dual volume $\widehat{V}_G(\cdot)$,

G increasing, $\widehat{V}_G(K) = \inf \{ \eta > 0 : \int_{S^{n-1}} G(\frac{\rho_K(u)}{\eta}, u) \, du \le 1 \}$

G decreasing, $\widehat{V}_G(K) = \inf \{ \eta > 0 : \int_{S^{n-1}} G(\frac{\rho_K(u)}{\eta}, u) \, du \ge 1 \}$.

Replace $\widetilde{V}_G(\cdot)$ by $V_G(\cdot)$ and $\widehat{V}_G(\cdot)$, we analyze cases for G increasing / decreasing, and inf / sup results.