A generalization of Bang's lemma

– GERGELY AMBRUS –

AGA seminar

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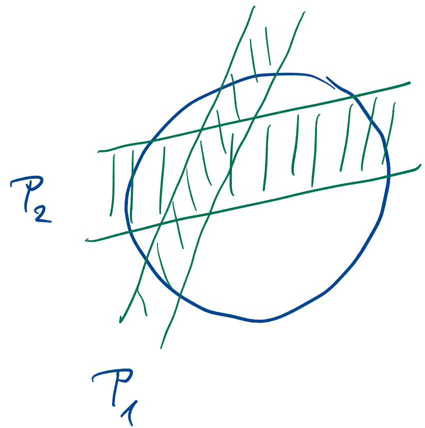
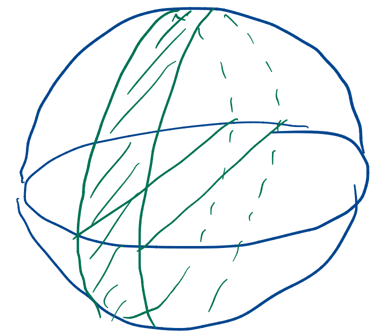
Plank problem 1932 (Tarski)



C unit disc



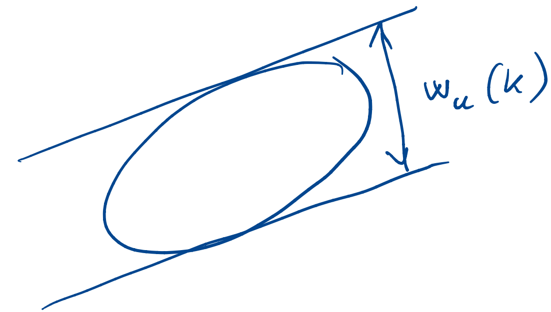
P_i planks

$C \subset \bigcup P_i$

$\Rightarrow \sum w(P_i) \geq 2$

$K \subset \mathcal{K}^d$ $\qquad$ $u \in S^{d-1}$

$$w_u(K)$$



$$w(K) = \min w_u(K)$$

Question: $\quad$ if $K \subset \bigcup P_i$

$$\sum w(P_i) \geq w(K) \quad ?$$

↳ Thm Bang (1951) $\quad$ ↳ this is true

Bang's lemma:

$u_1, \dots, u_n \in S^{d-1}$ $\quad |u_i| = 1$

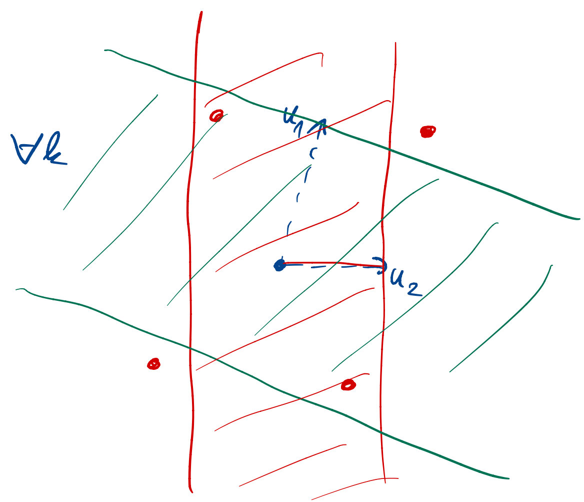
$w_1, \dots, w_n > 0$

$\forall \; t_1, \dots, t_n \in \mathbb{R} \quad \exists \, \varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$

s.t. $u := \sum_{1}^{n} \varepsilon_i u_i w_i$

$$|\langle u, u_k \rangle - t_k| \geq w_k \quad \forall k$$

inner product



Affine plank problem:

$$w_K(P) = \frac{w(P)}{w_u(K)}$$

u normal vector of P

$K \in \mathcal{K}$ $\qquad P_1, \dots, P_n$ cover K

<u>Conjecture</u>: $\displaystyle\sum_{i=1}^{n} w_K(P_i) \geq 1.$

(affine plank problem)

<u>K. Ball '91</u>: proved, if $K = -K$

two planks in $\mathbb{R}^2$

three — " —

only two normal directions.

⋮

Kadets 2005.

$K \in \mathcal{K}^d$ $\qquad K_1, \dots, K_n \in \mathcal{K}^d$

$$K \subset \bigcup_{i=1}^{n} K_i$$

$r(K) :=$ inradius of K

Thm: $\quad \underline{\sum r(K_i)} \geq r(K)$

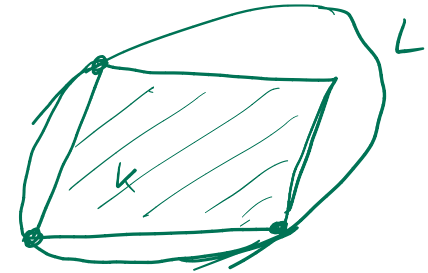
$\hookrightarrow$ based on a variant of Bang's lemma.

$K, L \in \mathcal{K}^d$ $\qquad$ (L closed, convex)

$$r_K(L) := \max\{\lambda : \lambda K + x \subset L \text{ for some } x \in \mathbb{R}^d\}$$

Bezdek ; Bezdek & Bezdek ;

Akopyan & Karasev



If K is covered by $K_1, \dots, K_n$, then (K_i) cannot be too small.

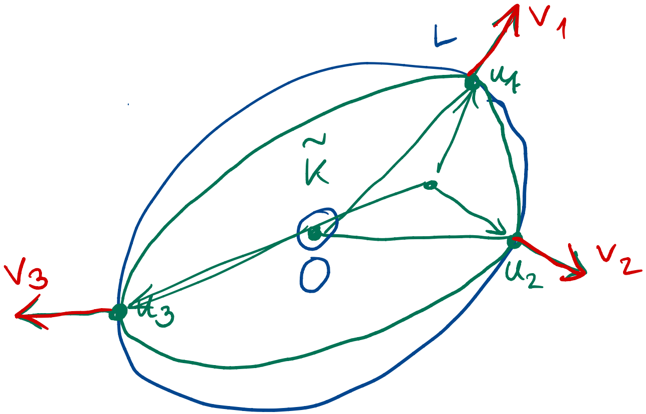
$$w_K(P) = r_K(P)$$

Affine plank: $K \subset \cup P_i$

then $\sum r_K(P_i) \geq 1$.

<u>Conjecture</u>: if $K, K_1, \dots, K_n \in \mathcal{K}^d$, and $K \subset \cup K_i$

then $\sum r_K(K_i) \geq 1$.



Contact pair:

$$(u_i, v_i)$$

s.t. $u_i \in \partial\tilde{K} \cap \partial L$

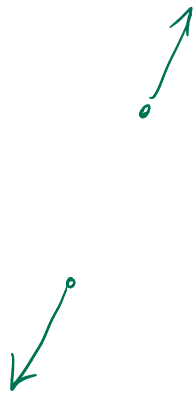
v_i is a common outer normal at u_i

Extend Bang's lemma to contact pairs which form complete systems

A set of contact pairs is <u>complete</u> if $O \in \operatorname{conv}(v_i)$

$\alpha_i \geq 0 \qquad \sum \alpha_i = 1$

$$\boxed{\begin{aligned} \sum \alpha_i v_i &= 0 \\ \sum \alpha_i u_i &= 0 \end{aligned}}$$



original Bang

$u_i = v_i$

complete set = two opposite vectors

$w = (u, v)$ $\qquad u, v \in \mathbb{R}^d$

$\widehat{\mathbb{R}^d \times \mathbb{R}^d}$

$\hat{w} = (v, u)$

Theorem. Assume $W_1, \dots, W_n \subset \mathbb{R}^d \times \mathbb{R}^d$, finite, $(0,0) \in \operatorname{conv} W_i$ $\forall i$.

Then $\forall z_1, \dots, z_n \in \mathbb{R}^d \times \mathbb{R}^d$ $\exists w_i \in W_i$ s.t. if $w = \sum w_i$

- $\langle w - z_k, \hat{w}_k \rangle \geq \langle w_k, \hat{w}_k \rangle$. $\quad \forall k = 1, \dots, n$

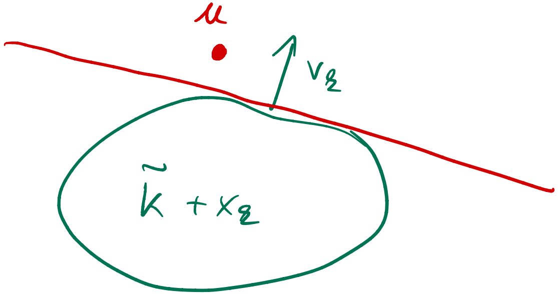
$w_i = (u_i, v_i)$ $\qquad w = (u, v)$ $\qquad u = \sum u_i$

$z_k = (x_k, y_k)$ $\qquad v = \sum v_i$

$$\boxed{\langle u - x_k, v_k \rangle} + \langle v - y_k, u_k \rangle \geq 2 \langle u_k, v_k \rangle.$$

point u is outside of the shifted copy of the homothet of K.

$\tilde{K} + x_k$



e.g. A symm.

$v_i = A u_i$

$$\boxed{\langle u - x_\varepsilon, v_\varepsilon \rangle} + \langle v - y_\varepsilon, u_\varepsilon \rangle \geq 2 \langle u_\varepsilon, v_\varepsilon \rangle .$$

Remark: if $\langle u_i, v_j \rangle = \langle u_j, v_i \rangle$

$$\langle u - x_\varepsilon, v_\varepsilon \rangle + \langle u, v_\varepsilon \rangle - \langle y_\varepsilon, u_\varepsilon \rangle$$

$$2 \langle u, v_\varepsilon \rangle$$

Do need symmetry for applications for covering results

$$\boxed{\langle u - x_\ell , v_\ell \rangle} + \langle v - y_\ell , u_\ell \rangle \geq 2 \langle u_\ell , v_\ell \rangle .$$

Proof. $u = \sum u_i \qquad v = \sum v_i$

Select (u_i, v_i) from W_i , $i = 1, \dots, n$ s.t

$$\sum_{i \neq j} \langle u_i , v_j \rangle - \sum_i \langle x_i , v_i \rangle - \sum_j \langle u_j , y_j \rangle \quad \text{is maximal}$$

$k \qquad (u_\ell , v_\ell) \in W_\ell$

$\exists \, \alpha(w_\ell') \geq 0 \qquad w_\ell' \in W_\ell \qquad \sum \alpha(w_\ell') = 1$

$\sum \alpha(w_\ell') \, (u_\ell', v_\ell') = (0, 0)$

Switch (u_ℓ, v_ℓ) to (u_ℓ', v_ℓ')

$$0 \geq \sum_{i \neq \xi} \langle u_i, v_\xi' - v_\xi \rangle + \sum_{j \neq \xi} \langle u_\xi' - u_\xi, v_j \rangle - \langle x_\xi, v_\xi' - v_\xi \rangle - \langle u_\xi' - u_\xi, y_\xi \rangle$$

$$\sum \quad \alpha(w_\xi')$$

$$0 \geq \sum_{i \neq \xi} \langle u_i, -v_\xi \rangle + \sum_{j \neq \xi} \langle -u_\xi, v_j \rangle + \langle x_\xi, v_\xi \rangle + \langle u_\xi, y_\xi \rangle$$

$$\sum_{i \neq \xi} \langle u_i, -v_\xi \rangle = \langle u, -v_\xi \rangle - \langle u_\xi, -v_\xi \rangle$$

$$\langle u, v_\xi \rangle + \langle u_\xi, v \rangle - \langle x_\xi, v_\xi \rangle - \langle u_\xi, y_\xi \rangle \geq 2 \langle u_\xi, v_\xi \rangle \quad \square$$

$$\langle u - x_\xi, v_\xi \rangle + \langle u_\xi, v - y_\xi \rangle \geq 2 \langle u_\xi, v_\xi \rangle.$$

Corollaries: if $0 \in \operatorname{conv} U_i \quad \forall i$

$\forall x_1, \dots, x_n \in \mathbb{R}^d \quad \exists u_i \in U_i$

$u = \sum u_i$

$$\langle u - x_k, u_k \rangle \geq |u_k|^2 \quad \forall k.$$

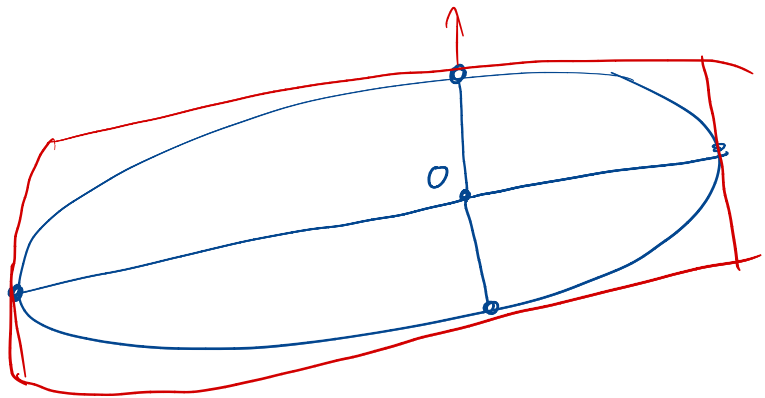
Thm. $r_K(K_i) \geq 1$ if $K, K_1, \dots, K_n \in \mathcal{K}^d$

$\forall i \quad \exists o_i \quad r_K(K_i) \cdot K - o_i$ and $K_i - o_i$

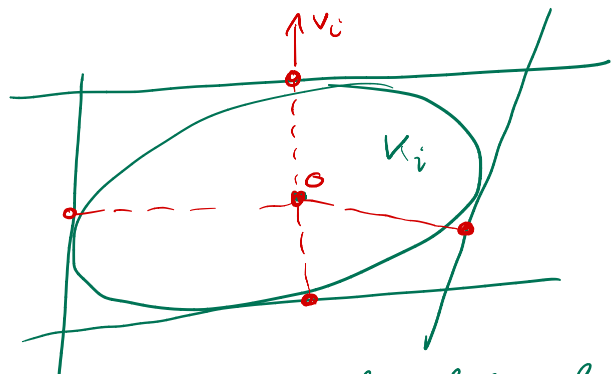
have complete set of contact pairs s.t.

$$\langle u_i, v_j \rangle = \langle u_j, v_i \rangle \quad \forall i \neq j.$$

Special case : contact pairs of the form (u,u)



K



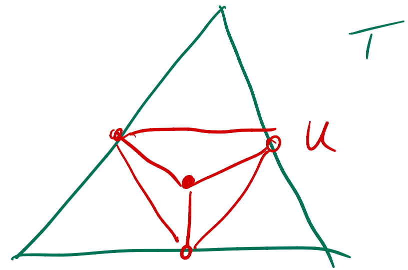
projections

$\to u_i$

Claim : $u_1 + \dots + u_n$

No set of shifts of K_i's which cover $u_1 + \dots + u_n$

Claim.



$$\operatorname{conv} U = -\frac{1}{d} \cdot T$$

Thm. Assume $\lambda_1, \dots, \lambda_n > 0$ s.t.

$-\lambda_1 T, \dots, -\lambda_n T$ have shifts which cover T

Then $\sum \lambda_i \geq d$.

Conjecture of Soltan; G. Fejes Tóth $K \in \mathcal{K}^d$

Assume $\lambda_i \in (-1, 1)$ s.t. $\lambda_1 K, \dots, \lambda_n K$ have shifts covering K. Then $\sum |\lambda_i| \geq d$.