"ISOPERIMETRIC SETS FOR WEIGHTED TWISTED EINGENVALUES"

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DIRICHLET BOUNDARY CONDITIONS:

THE FABER-KRAHN INEQUALITY

$\Omega \subseteq \mathbb{R}^n$ bounded, open set

$$\begin{cases} -\Delta u = \lambda u & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

$$\lambda_1^D(\Omega) = \min\left\{ \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega \phi^2 dx} ; \phi \in H_0^1(\Omega) \setminus \{0\} \right\}$$

Faber-Krahn inequality:

$$\lambda_1^D(\Omega) |\Omega|^{\frac{2}{n}} \geq \lambda_1^D(B) |B|^{\frac{2}{n}}$$

equality holding iff Ω is a ball.

Q: Is it possible to derive similar lower bounds for the first eigenvalue of the Laplacian defined in other subsets of $H^1(\Omega)$?

NEUMANN BOUNDARY CONDITIONS

$\Omega \subseteq \mathbb{R}^n$ bounded, open set

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

$\mu_1(\Omega) = 0$, with constant corresponding eigenfunctions

$$\mu_2(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla \phi|^2 dx}{\int_\Omega \phi^2 dx} : \begin{array}{l} \phi \in H^1(\Omega) \setminus \{0\}, \\ \int_\Omega \phi \, dx = 0 \end{array} \right\}$$

$\mu_2(\Omega)$ cannot be bounded from below in terms of $|\Omega|$:

if Ω has two connected components, $\mu_2(\Omega) = 0$ while $|\Omega| > 0$.

NEUMANN BOUNDARY CONDITIONS:

THE SZEGŐ - WEINBERGER INEQUALITY

$$\mu_2(\Omega)\,|\Omega|^{\frac{2}{n}} \le \mu_2(B)\,|B|^{\frac{2}{n}}$$

equality holding iff Ω is a ball.

$\Downarrow$

We can hope to gain a universal lower bound by considering boundary conditions further than zero-mean value.

$\Downarrow$

TWISTED EINGENVALUE

TWISTED EIGENVALUE PROBLEM

$\Omega \subseteq \mathbb{R}^n$ bounded, open set

$$(P)\begin{cases} -\Delta u = \lambda u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \Delta u \, dx & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\rightarrow$ test $\phi \in H_0^1(\Omega)$

or, equivalently,

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\rightarrow$ test $\phi \in H_0^1(\Omega)$, $\displaystyle\int_\Omega \phi \, dx = 0$.

- This eigenvalue problem has been first introduced by Barbosa - Berard (2000) in the context of constant mean curvature surfaces.

TWISTED EIGENVALUE PROBLEM

- Among other results, B-B proved that problem (P) has a discrete spectrum: $0 < \lambda_1^T(\Omega) \leq \lambda_2^T(\Omega) \leq \ldots$
and

$$\lambda_1^T(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla \phi|^2 dx}{\int_\Omega \phi^2 dx} : \phi \in H_0^1(\Omega) \setminus \{0\}, \int_\Omega \phi \, dx = 0 \right\}$$

- Moreover, the spectra of the Dirichlet and the twisted problems are intertwined:

$$0 < \lambda_1^D(\Omega) < \lambda_1^T(\Omega) \leq \lambda_2^D(\Omega) \leq \lambda_2^T(\Omega) \leq \ldots$$

and a Courant type theorem holds true for nodal domains of the twisted eigenfunctions:

u_k eigenfunction corresponding to $\lambda_k^T(\Omega)$ $(k \geq 1)$

$\Rightarrow$ u_k has at most $k+1$ nodal domains.

TWISTED EIGENVALUE PROBLEM

- Euler - Lagrange equation:

$$(P)\begin{cases} -\Delta u = \lambda_1^T(\Omega)\, u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \Delta u \, dx & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\to$ test $\phi \in H_0^1(\Omega)$

or, equivalently,

$$\begin{cases} -\Delta u = \lambda_1^T(\Omega)\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\to$ test $\phi \in H_0^1(\Omega)$, $\int_\Omega \phi \, dx = 0$.

<u>Remark</u>. Due to the presence of the average of the Laplacian, problems like (P) are often referred to as nonlocal eigenvalue problems (B., Freitas, Henrot, Mitsch, Trombetti C., Mazarov, Dacorogna, Croce, Pisante, Gangbo, Subia, Buslaev, ...)

EXAMPLES (B.-B. 2000)

1) $\Omega = [0,a] \subset \mathbb{R}$ $\boxed{n=1}$

$$\begin{cases} -u'' = \lambda u - \frac{1}{a}\int_0^a u''(x)\,dx \\ \\ u(0) = u(a) = 0 \\ \int_0^a u(x)\,dx = 0 \end{cases}$$

$\to$ Eigenvalues

$$\frac{4k^2\pi^2}{a^2}$$

Eigenfunctions

$$\sin\left(\frac{2k\pi t}{a}\right)$$

$$\frac{4\tau_m^2}{a^2}$$

$$\cos\left(\frac{2\tau_m t}{a} - \tau_m\right) - \tau_m$$

$k, m \in \mathbb{N}$, τ_m zeros of $f(t) = t - \operatorname{tg} t$.

$$\lambda_1^D < \lambda_1^T = \lambda_2^D < \lambda_2^T < \lambda_3^D < \lambda_3^T = \lambda_4^D < \lambda_4^T < \lambda_5^D \dots$$

2) $\Omega = B(0,1) \subset \mathbb{R}^2$ $\boxed{n=2}$

The Dirichlet and twisted spectra can only differ for eigenvalues corresponding to rotation invariant eigenfunctions.

→ Eigenfunctions	Eigenvalues	
$J_0(j_{2,m}\lvert x\rvert) - J_0(j_{2,m})$	$j_{2,m}^2$	$m \geq 1$
$J_1(j_{1,m}\lvert x\rvert)\cos\vartheta$	$j_{1,m}^2$	
$J_1(j_{1,m}\lvert x\rvert)\sin\vartheta$	$j_{1,m}^2$	
$J_2(j_{2,m}\lvert x\rvert)\cos 2\vartheta$	$j_{2,m}^2$	
$J_2(j_{2,m}\lvert x\rvert)\sin 2\vartheta$	$j_{2,m}^2$	
$\vdots$	$\vdots$	

$$\lambda_1^D < \underbrace{\lambda_1^T = \lambda_2^D = \lambda_2^T = \lambda_3^D}_{\text{multiplicity } 2} < \underbrace{\lambda_3^T = \lambda_4^D = \lambda_4^T = \lambda_5^D = \lambda_5^T}_{\text{multiplicity } 3} < \lambda_6^D$$

$\vdots$

A SHAPE OPTIMIZATION PROBLEM

In Freitas - Henrot (2004), the authors prove:

$\Omega \subseteq \mathbb{R}^n$ bounded, open set

$$\lambda_1^T(\Omega) \geq \lambda_1^T(B_1 \cup B_2)$$

where B_1, B_2 are two disjoint balls with measure equal to $\frac{|\Omega|}{2}$, equality holding (for regular Ω) iff $\Omega = B_1 \cup B_2$.

<u>Remark.</u>

It is clear that the eigenvalue does not change if we add or remove sets having zero capacity. This is the reason why they need to consider regular (e.g. Lipschitzian) sets to investigate the equality case.

A SHAPE OPTIMIZATION PROBLEM

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Remark.

Since the first eigenfunction u_1 changes sign in Ω, the above estimate is more related to the Krahn-Szegö inequality, stating that

$$\lambda_2^D(\Omega)\,|\Omega|^{2/n} \geq \underbrace{\lambda_1(B)}_{\shortparallel}\, 2^{2/n}\, |B|^{2/n}$$

the first eigenvalue of the union of two identical balls, which has multiplicity 2 and coincides with the first eigenvalue of one of the two balls.

Remark.

An analogous result is proved by Greco-Lucia (2008), when u satisfies constant Dirichlet data:

$$\lambda^T_{1,c}(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega \phi^2 dx} : \phi - c \in H^1_0(\Omega), \int_\Omega \phi \, dx = 0 \right\}.$$

$(c \in \mathbb{R})$

Remark.
It is not a case that in all the above inequalities balls play a fundamental role. Balls are optimal sets in the classical isoperimetric inequality:

$$\frac{P(\Omega)}{|\Omega|^{1-\frac{1}{n}}} \geq \frac{P(B)}{|B|^{1-\frac{1}{n}}} \quad !!$$

A NATURAL QUESTION

$\Omega \subseteq \mathbb{R}^n$, Ω of locally finite perimeter

$\gamma(x)$ positive function on $\mathbb{R}^n$ DENSITY

$$P_\gamma(\Omega) = \int_{\partial\Omega} \gamma(x)\, d\mathcal{H}^{n-1}, \quad |\Omega|_\gamma = \int_\Omega \gamma(x)\, dx$$

Assume that the isoperimetric problem is solved for any $m>0$, that is there exists Ω^* such that

$$P_\gamma(\Omega^*) = \min\left\{ P_\gamma(A) : A \subseteq \mathbb{R}^n,\ A \text{ of locally finite perimeter},\ |A|_\gamma = m \right\}$$

$\Updownarrow$

$$\boxed{P_\gamma(\Omega) \geq P_\gamma(\Omega^*), \qquad |\Omega|_\gamma = |\Omega^*|_\gamma .}$$

A NATURAL QUESTION

$\Omega \subset \mathbb{R}^n$ open set

We consider the following weighted twisted eigenvalue problem:

$$\begin{cases} -\operatorname{div}(\gamma(x)\nabla u) = \gamma(x)\lambda u - \dfrac{\gamma(x)}{|\Omega|_\gamma}\displaystyle\int_\Omega \operatorname{div}(\gamma(x)\nabla u)\,dx & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases}$$

and we focus our attention on the first eigenvalue

$$\lambda_{1,\gamma}^T(\Omega) = \min\left\{ \frac{\displaystyle\int_\Omega |\nabla\phi|^2 d\gamma}{\displaystyle\int_\Omega \phi^2 d\gamma} : \phi \in H_0^1(\Omega, d\gamma)\setminus\{0\},\ \int_\Omega \phi\, d\gamma = 0 \right\}$$

$\underline{Q}$: Are there sufficient conditions on γ in order to have

$$\lambda_{1,\gamma}^T(\Omega) \geq \lambda_{1,\gamma}^T(\Omega_1^* \cup \Omega_2^*), \quad \text{with } \Omega_{1,2}^*$$

isoperimetric sets having $|\Omega_1^*|_\gamma = |\Omega_2^*|_\gamma = \dfrac{|\Omega|_\gamma}{2}$?

ASSUMPTIONS ON γ

1) $H_0^1(\Omega, d\gamma) \hookrightarrow\hookrightarrow L^2(\Omega, d\gamma)$

(This immediately implies the variational characterization of $\lambda_{1,\gamma}^T(\Omega)$!)

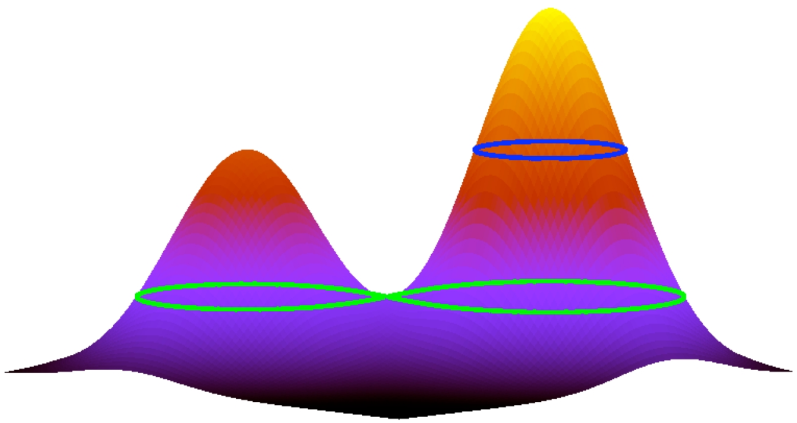
2) A Pólya-Szegö type inequality holds true:

$$\int_\Omega |\nabla u|^2 \, d\gamma \geq \int_{\Omega^\star} |\nabla u^\star|^2 \, d\gamma ,$$

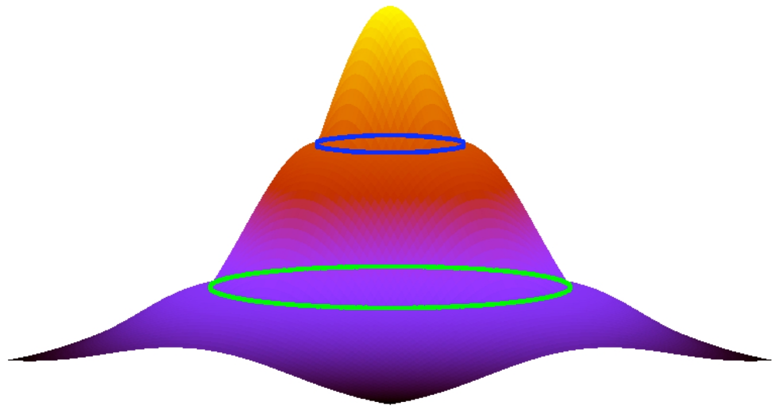
for any $u \in H_0^1(\Omega, d\gamma)$

where
- $\Omega^\star$ is an isoperimetric set s.t. $|\Omega|_\gamma = |\Omega^\star|_\gamma$
- $u^\star$ is the weighted symmetric rearrangement of u, i.e. $u^\star$ is the only function defined in $\Omega^\star$ whose superlevel sets are isoperimetric sets having the same weighted measure as the corresponding superlevel sets of u.

$\gamma(x) \equiv 1$, u^* Schwarz symmetrand of u



$u(x)$



$u^\star(x)$

ASSUMPTIONS ON γ AND EXAMPLES

3) If we fix $m_1, m_2 > 0$, there exist two isoperimetric sets Ω_1^*, Ω_2^* such that

$$|\Omega_1^*|_\gamma = m_1, \quad |\Omega_2^*|_\gamma = m_2 \quad \text{and} \quad \Omega_1^* \cap \Omega_2^* = \emptyset .$$

Examples.

1) $\gamma(x) \equiv 1 \to$ Classical Laplacian operator
$\to$ Isoperimetric sets are balls

2) $\gamma(x) = \pi^{-\frac{n}{2}} e^{-|x|^2} \to$ Hermite operator
$\to$ Isoperimetric sets are half spaces

[Sudakov-Tsirelson 74, Borell 75, Ehrhard 84, Carlen-Kerce 01]

3) $\gamma(x) = x_n^k \quad (k \geq 0)$

$\to$ Weighted operator with ellipticity x_n^k
$\to$ Isoperimetric sets are half balls centered on $\{x_n = 0\}$

[Maderna-Salsa 81, Brock-Chiacchio-Mercaldo 12]

EXAMPLES

Remark.

We stress that in cases 1) and 3) we work with bounded sets, while in case 2) unbounded sets are allowed (remember that the Gaussian density is a probability density, that is

$$|\mathbb{R}^n|_\gamma = 1 < \infty \, !$$)

4) The densities

$$\gamma_1(x) = e^{|x|^2}, \quad \gamma_2(x) = e^{g(|x|)} \quad (g \text{ convex})$$

$$\gamma_3(x) = x_n^k \, e^{|x|^2} \quad (k \geq 0)$$

do not satisfy the third condition.

Indeed, isoperimetric sets for $\gamma_{1,2}$ are balls centered at the origin, while isoperimetric sets for γ_3 are half-balls contained in the half space $\{x_n > 0\}$, centered at the origin.

[Boyer - Cañete - Morgan - Rosales 08, Figalli - Maggi 13, Cabré - Ros Oton - Serra 12, Chambers 19, Alvino - Brock - Chiacchio - Mercaldo - Posteraro 17, 19, 21]

PROPERTIES OF $\lambda^T_{1,\gamma}(\Omega)$

1) A positive number λ is an eigenvalue for both the Dirichlet and the twisted problems iff there exists an associated eigenfunction for the Dirichlet problem having zero avarage.

2) Denoted by u_1 an eigenfunction corresponding to $\lambda^T_{1,\gamma}(\Omega)$, u_1 has precisely two nodal domains Ω^+, Ω^- and

$$\min\left\{\lambda^D_{1,\gamma}(\Omega^+), \lambda^D_{1,\gamma}(\Omega^-)\right\} \leq \lambda^T_{1,\gamma}(\Omega) \leq$$

$$\leq \max\left\{\lambda^D_{1,\gamma}(\Omega^+), \lambda^D_{1,\gamma}(\Omega^-)\right\}$$

Moreover, u_1 cannot vanish on any connected component of Ω.

3) $\lambda^D_{k,\gamma}(\Omega) \leq \lambda^T_{k,\gamma}(\Omega) \leq \lambda^D_{k+1,\gamma}(\Omega)$, $\quad k \geq 1$.

4) $\lambda^T_{1,\gamma}(\Omega)$ is monotone decreasing with respect to the inclusion of sets.

A FIRST STEP TOWARD: $\lambda_{1,\gamma}^T(\Omega) \geq \lambda_{1,\gamma}^T(\Omega_1^* \cup \Omega_2^*)$

Let u_1 be an eigenfunction corresponding to $\lambda_{1,\gamma}^T(\Omega)$

Set

$$\Omega_+ = \{x \in \Omega : u_1(x) > 0\}, \quad \Omega_- = \{x \in \Omega : u_1(x) < 0\},$$

Ω_+^*, Ω_-^* disjoint isoperimetric sets

s.t. $|\Omega_+^*|_\gamma = |\Omega_+|_\gamma$

$|\Omega_-^*|_\gamma = |\Omega_-|_\gamma$.

Prop.

$$\boxed{\lambda_{1,\gamma}^T(\Omega) \geq \lambda_{1,\gamma}^T(\Omega_+^* \cup \Omega_-^*).}$$

Pf.: Symmetrization arguments

Pólya - Szegö principle

$\Rightarrow$ We can reduce ourselves to minimize $\lambda_{1,\gamma}^T$ among unions of disjoint isoperimetric sets, having the same measure as Ω.

SOME DETAILS OF THE PROOF

Let u be an eigenfunction corresponding to $\lambda^T_{1,\gamma}(\Omega)$:

$$\Omega_+ = \{u > 0\}, \quad \Omega_- = \{u < 0\}$$

$$\lambda^T_{1,\gamma}(\Omega) = \frac{\int_\Omega |\nabla u|^2 d\gamma}{\int_\Omega (u)^2 d\gamma} =$$

$$= \frac{\int_{\Omega_+} |\nabla u_+|^2 d\gamma + \int_{\Omega_-} |\nabla u_-|^2 d\gamma}{\int_{\Omega_+} (u_+)^2 d\gamma + \int_{\Omega_-} (u_-)^2 d\gamma}$$

$$\geq \frac{\int_{\Omega_+^*} |\nabla u_+^*|^2 d\gamma + \int_{\Omega_-^*} |\nabla u_-^*|^2 d\gamma}{\int_{\Omega_+^*} (u_+^*)^2 d\gamma + \int_{\Omega_-^*} (u_-^*)^2 d\gamma}$$

and

$$\int_{\Omega_+^*} (u_+)^* d\gamma - \int_{\Omega_-^*} (u_-)^* d\gamma = \int_{\Omega_+} u_+ \, d\gamma + \int_{\Omega_-} u_- \, d\gamma = 0 .$$

Hence : (2)

$$\lambda_{1,\gamma}^{T}(\Omega) \geq \frac{\int_{\Omega_+^*} |\nabla u_+^*|^2 d\gamma + \int_{\Omega_-^*} |\nabla u_-^*|^2 d\gamma}{\int_{\Omega_+^*} (u_+^*)^2 d\gamma + \int_{\Omega_-^*} (u_-^*)^2 d\gamma}$$

with $\int_{\Omega_+^*} u_+^* \, d\gamma = \int_{\Omega_-^*} u_-^* \, d\gamma$.

This implies that

$$\lambda_{1,\gamma}^{T}(\Omega) \geq \inf \left\{ \frac{\int_{\Omega_+^*} |\nabla f|^2 d\gamma + \int_{\Omega_-^*} |\nabla g|^2 d\gamma}{\int_{\Omega_+^*} f^2 d\gamma + \int_{\Omega_-^*} g^2 d\gamma} : f \in H_0^1(\Omega_+^*, d\gamma),\ g \in H_0^1(\Omega_-^*, d\gamma),\ \int_{\Omega_+^*} f \, d\gamma = \int_{\Omega_-^*} g \, d\gamma \right\}$$

$$= \lambda_\gamma^* .$$

Standard arguments in Calculus of Variations ensure that the infimum is attained by a couple

$$(f_+, f_-) \in H_0^1(\Omega_+^*, d\gamma) \times H_0^1(\Omega_-^*, d\gamma).$$

By writing the Euler-Lagrange equation associated to λ_γ^*, we get that

$$w = \begin{cases} f^+ & \text{in } \Omega_+^* \\ -f^- & \text{in } \Omega_-^* \end{cases}$$

solves the eigenvalue problem

$$\begin{cases} -\operatorname{div}\left(\gamma(x)\nabla w\right) = \lambda_\gamma^* \gamma(x) w - \gamma(x)\left(\dfrac{1}{|\Omega|_\gamma}\displaystyle\int_\Omega \operatorname{div}(\gamma(x)\nabla w)\right) & \text{in } \Omega_+^* \cup \Omega_-^* \\ w = 0 & \text{on } \partial(\Omega_+^* \cup \Omega_-^*), \end{cases}$$

or, equivalently, λ_γ^* is a twisted eigenvalue on the union $\Omega_+^* \cup \Omega_-^*$. Finally

$$\lambda_{1,\gamma}^T(\Omega) \geq \lambda_\gamma^* \geq \lambda_{1,\gamma}^T(\Omega_+^* \cup \Omega_-^*).$$

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A SECOND STEP TOWARD: $\lambda_{1,\gamma}^{T}(\Omega) \geq \lambda_{1,\gamma}^{T}(\Omega_1^* \cup \Omega_2^*)$

<u>Prop</u>:

Assume that the optimal set $\tilde{\Omega}$ for the problem

$$\min \left\{ \lambda_{1,\gamma}^{T}(\Omega) \; : \; |\Omega|_{\gamma} = m \right\}$$

is smooth. Denoted by $\tilde{u}_1$ an eigenfunction corre-sponding to $\lambda_{1,\gamma}^{T}(\tilde{\Omega})$, it holds

$$\boxed{|\nabla \tilde{u}_1|^2 = \text{constant} \quad \text{on } \partial\tilde{\Omega}.}$$

<u>PF</u>: Shape derivative arguments:

$$(\lambda_{1,\gamma}^{T})'(\tilde{\Omega}) = -\int_{\partial\tilde{\Omega}} \left(\frac{\partial \tilde{u}_1}{\partial \nu}\right)^2 V \cdot \nu \, \gamma(x) \, d\mathcal{H}^{n-1}$$

where $V: \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field.
Being $\tilde{\Omega}$ optimal, we get

$$(\lambda_{1,\gamma}^{T})'(\tilde{\Omega}) = 0 \quad \Rightarrow \quad |\nabla \tilde{u}_1|^2 = \text{constant on } \partial\tilde{\Omega}.$$

\#

THE MAIN STEP

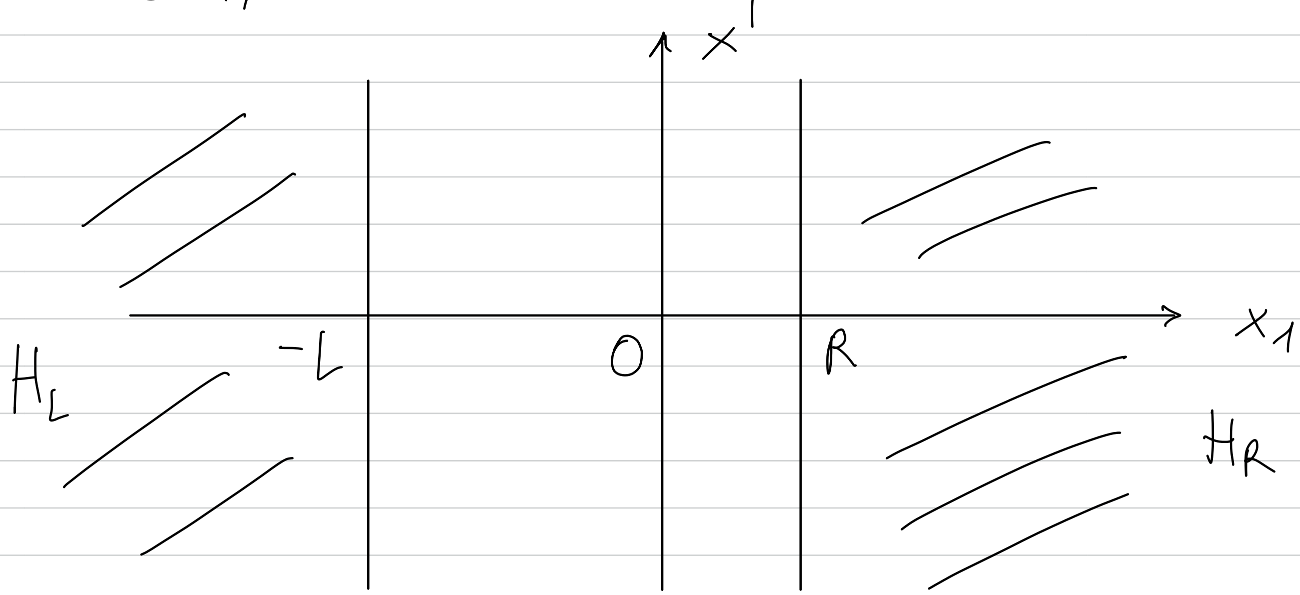
We will focus our attention on the examples:

- $\gamma(x) = \pi^{-\frac{n}{2}} e^{-|x|^2}$ Gaussian density

- $\gamma(x) = x_n^k \quad (k > 0)$ Power density

THE MAIN STEP : $\gamma(x) = \pi^{-\frac{n}{2}} e^{-|x|^2}$

$x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$



$H_L = \{ x \in \mathbb{R}^n \ / \ x_1 < -L \}$

$H_R = \{ x \in \mathbb{R}^n \ / \ x_1 > R \}$

$\mathcal{H} = \{ H_L \cup H_R \ : \ |H_L|_\gamma = |H_R|_\gamma \ , \ |H_L \cup H_R|_\gamma = |\Omega|_\gamma \}$

An eigenfunction u_1 corresponding to $\lambda^T_{1,\gamma}(H_L \cup H_R)$ can be written as

$$u(x) = \begin{cases} u_L(x_1) & \text{in } H_L \\ u_R(x_1) & \text{in } H_R \end{cases}$$

where u_L, u_R respectively solve

$$\begin{cases} y'' - 2x_1 y' + \lambda_{1,\gamma}^T (H_L \cup H_R) y = c, & x_1 < -L \\ y \in H_0^1((-\infty, -L), d\gamma_1) \end{cases}$$

and

$$\begin{cases} y'' - 2x_1 y' + \lambda_{1,\gamma}^T (H_L \cup H_R) y = c, & x_1 > R \\ y \in H_0^1((R, +\infty), d\gamma_1) . \end{cases}$$

$$\left(\gamma_1 = \frac{e^{-x_1^2}}{\sqrt{2\pi}} \right)$$

It can be easily checked that

$$u = \begin{cases} u_L(x_1) = A\left(H_\nu(-x_1) - H_\nu(L)\right) & , x_1 < -L \\ u_R(x_1) = B\left(H_\nu(R) - H_\nu(x_1)\right) & , x_1 > R, \end{cases}$$

where H_ν is the Hermite function with index ν, being $\boxed{2\nu = \lambda_{1,\gamma}^T (H_L \cup H_R)}$ and, since the constant c is the same in both the equations,

$$A H_\nu(L) + B H_\nu(R) = 0.$$

CLAIM : $R = L$ for the optimal set $\tilde{\Omega}$

Suppose by contradiction that $R < L$ and let us consider the following transformation:

$$R \to R + \varepsilon$$
$$L \to L^{\varepsilon} = L - \frac{e^{-R^2}}{e^{-L^2}} \varepsilon + o(\varepsilon).$$

this factor ensures that $|H_{R+\varepsilon} \cup H_{L^{\varepsilon}}|_{\gamma} = |\Omega|_{\gamma}$!

$$\Rightarrow \left(\lambda_{1,\gamma}^{T}\right)' = e^{-R^2}\left(u_R'(R)^2 - u_L'(L)^2\right) =$$

$$= 4\nu^2 A^2 H_\nu(L)^2 \left[\left(\frac{H_{\nu-1}(R)}{H_\nu(R)}\right)^2 - \left(\frac{H_{\nu-1}(L)}{H_\nu(L)}\right)^2\right].$$

Using some known properties of H_ν (number of zeros, recurrence relations, Turan's inequality) we get that

$$u'(R)^2 - u'(L)^2 > 0 \quad \lightning \quad |\nabla \tilde{u}|^2 = \text{constant on } \partial\tilde{\Omega}$$

DETAILS:

- $$\varphi_\nu(x_1) = \frac{H_{\nu-1}(x_1)}{H_\nu(x_1)}, \quad x_1 > R.$$

φ_ν is well-defined: H_ν has a finite number of zeros and R is greater than the largest zero of H_ν.

- $$\varphi'_\nu(x_1) = \frac{H'_{\nu-1} H_\nu - H_{\nu-1} H'_\nu}{H_\nu^2} = \qquad \longleftarrow \qquad H'_\nu = 2\nu H_{\nu-1}$$

$$= \frac{2(\nu-1) H_{\nu-2} H_\nu - 2\nu H_{\nu-1}^2}{H_\nu^2}$$

$\nu - 1 < 0$: $\varphi'_\nu < 0$

$\nu - 1 > 0$: $H_\nu^2 - H_{\nu-1} H_\nu > 0$ implies $\varphi'_\nu < 0$

$\Downarrow$

φ_ν is strictly decreasing in $[R, +\infty[$

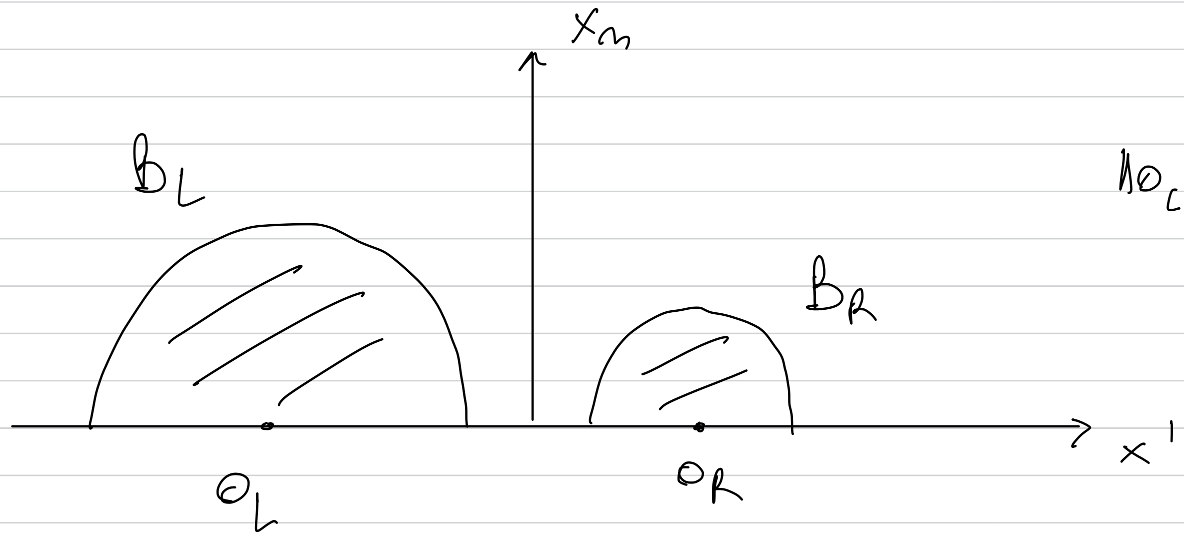
- $H_\nu(x_1) \sim x_1^\nu$ as $x_1 \to +\infty$ ensures that $\varphi_\nu(x_1) \to 0$ as $x_1 \to +\infty$

$\Downarrow$

$\varphi_\nu > 0$ in $[R, +\infty[$.

THE MAIN STEP: $\gamma(x) = x_n^k \quad (k \geq 0)$

$x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$



$$\mathcal{B} = \left\{ B_L \cup B_R : |B_L|_\gamma \geq |B_R|_\gamma , \ |B_L \cup B_R|_\gamma = |\Omega|_\gamma \right\}$$

An eigenfunction u_1 corresponding to $\lambda^T_{1,\gamma}(B_L \cup B_R)$ can be written as

$$u(x) = \begin{cases} u_L(\|x - O_L\|) & x \in B_L \\ u_R(\|x - O_R\|) & x \in B_R , \end{cases}$$

where u_L, u_R respectively solve

$$\begin{cases} r y'' + (n+k-1) y' + \lambda^T_{1,\gamma}(B_L \cup B_R)\, y = e r , & r \in (0, L) \\ y \in H^1_0(B_L, d\gamma) \end{cases}$$

and

$$\begin{cases} r y'' + (n+k-1)y' + \lambda_{1,\gamma}^{T}(B_L \cup B_R)\, y = c r, & r \in (0,R) \\ y \in H_0^1(B_R, d_\gamma). \end{cases}$$

It can be easily checked that

$$u = \begin{cases} u_L(r) = A\left(r^{\alpha} J_{-\alpha}(\lambda r) - L^{\alpha} J_{-\alpha}(\lambda L)\right), & r \in [0,L] \\ u_R(r) = B\left(r^{\alpha} J_{-\alpha}(\lambda r) - R^{\alpha} J_{-\alpha}(\lambda R)\right), & r \in (0,R], \end{cases}$$

where $J_{-\alpha}$ is the Bessel function of order

$$-\alpha = \frac{n+k}{2} - 1$$

of the first kind, $\boxed{\lambda^2 = \lambda_{1,\gamma}^{T}(B_L \cup B_R)}$

and, since the costant c is the same in both the equations,

$$A L^{\alpha} J_{-\alpha}(\lambda L) + B R^{\alpha} J_{-\alpha}(\lambda R) = 0.$$

CLAIM: $R = L$ for the optimal set $\tilde{\Omega}$

Suppose by contradiction that $R < L$ and let us consider the following transformation:

$$R \to R + \varepsilon$$
$$L \to L^{\varepsilon} = L - \left(\frac{R}{L}\right)^{n+k-1} \varepsilon + o(\varepsilon)$$

this factor ensures that $|B_{R+\varepsilon} \cup B_{L^\varepsilon}|_\gamma = |\Omega|_\gamma$!

$$\Rightarrow \left(\lambda^T_{1,\gamma}\right)' = \left(\frac{R}{L}\right)^{n+k-1} \left(\left(\frac{\partial u_L}{\partial r}(L)\right)^2 - \left(\frac{\partial u_R}{\partial r}(R)\right)^2 \right) =$$

$$= A^2 L^{2\alpha} J_{-\alpha}(\lambda L)^2 \left(\left(\frac{\alpha}{L} + \lambda \frac{J'_{-\alpha}(\lambda L)}{J_{-\alpha}(\lambda L)}\right)^2 - \left(\frac{\alpha}{R} + \lambda \frac{J'_{-\alpha}(\lambda R)}{J_{\alpha}(\lambda R)}\right)^2 \right)$$

Using some known properties of J_α (positive zeros, recurrence relations, Turan's inequality) we get that

$$\left(\frac{\partial u_L}{\partial r}(L)\right)^2 - \left(\frac{\partial u_R}{\partial r}(R)\right)^2 > 0 \; ↯$$

$|\nabla \tilde{u}|^2 =$ constant on $\partial \tilde{\Omega}$

TWO FINAL REMARKS

- If $k=0 \iff \gamma(x)\equiv 1$, we have an alternative proof of the already cited result by Freitas-Henrot for the first twisted eigenvalue of the Laplacian operator.

- The question of the existence of the optimal set for the problem

$$\min\left\{ \lambda_{1,\gamma}^{T}(A) : A \subseteq \mathbb{R}^n \text{ quasi-open}, \ |A|_\gamma = m \right\}$$

can be faced by using arguments from γ-convergence.

[Buttazzo-Dal Maso 93, Bucur-Buttazzo-Oudet 09, Mazzoleni-Pratelli 13, B.-Chiacchio-Henrot-Trombetti 15]

Thank you for

your attention ! !