Mean inequalities for symmetrizations of convex bodies

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(joint work with René Brandenberg, and Bernardo González Merino)

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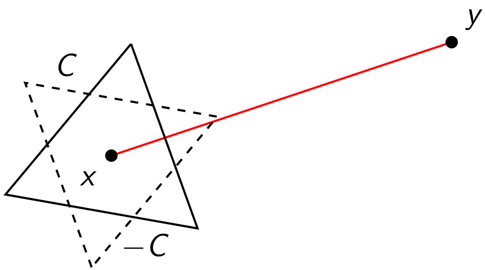
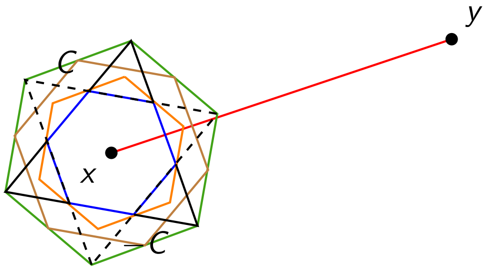
Motivation: Average Length


y
C
x
−C



$$\max\{\|x-y\|_C, \|y-x\|_C\} = \|x-y\|_{C\cap(-C)}$$

$$1/2\,(\|x-y\|_C + \|y-x\|_C) = \|x-y\|_{\left(\frac{C^\circ - C^\circ}{2}\right)^\circ}$$

$$R(\{x,y\},C) = \|x-y\|_{\frac{C-C}{2}}$$

$$\|x-y\|_{conv(C\cup(-C))}$$

AM-HM mean inequality

Let $a, b > 0$. Then

$$\min\{a, b\} \leq \frac{2ab}{a+b} \leq \frac{a+b}{2} \leq \max\{a, b\}$$

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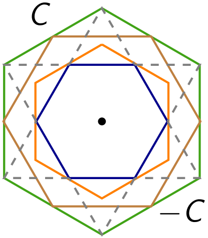
Let A, B be convex sets with $0 \in \mathrm{int}(A) \cap \mathrm{int}(B)$. Firey '61 has shown

$$A \cap B \subset \left(\frac{A^\circ + B^\circ}{2}\right)^\circ \subset \frac{A+B}{2} \subset \mathrm{conv}(A \cup B).$$

Always Optimal

Let C be a convex set with $0 \in \mathrm{int}(C)$. Then

$$C \cap (-C) \subset^{opt} \left(\frac{C^\circ + (-C)^\circ}{2} \right)^\circ \subset \frac{C + (-C)}{2} \subset^{opt} \mathrm{conv}(C \cup (-C)).$$



$$C \cap (-C) \subset^{opt} \left(\frac{C^\circ + (-C)^\circ}{2}\right)^\circ \subset \frac{C + (-C)}{2} \subset^{opt} \operatorname{conv}(C \cup (-C)).$$

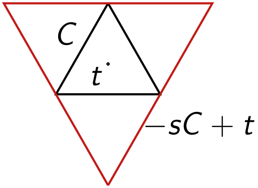
What about:

$$\left(\frac{C^\circ + (-C)^\circ}{2}\right)^\circ \subset^? \frac{C + (-C)}{2}$$

$$C \cap (-C) \subset^? \mathrm{conv}(C \cup (-C))$$

Minkowski Asymmetry

$$s(C) := inf\{\lambda > 0 | C \subset \lambda(-C) + t \quad \text{for some} \quad t \in \mathbb{R}\} = R(C, -C).$$



c = Minkowski center: $C - c \subset s(C)(-(C - c))$.
0 = Minkowski center: C is Minkowski centered.

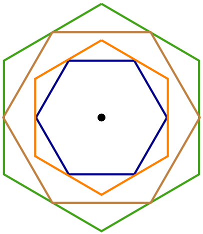
Theorem

Let S be a Minkowski centered regular n-simplex. Then

(i) $S \cap (-S) \subset^{opt} \operatorname{conv}(S \cup (-S))$, *if n is odd,*

(ii) $S \cap (-S) \subset^{opt} \frac{n}{n+1}\operatorname{conv}(S \cup (-S))$, *if n is even, and*

(iii) $\left(\frac{S^\circ - S^\circ}{2}\right)^\circ \subset^{opt} \frac{n(n+2)}{(n+1)^2} \cdot \frac{S-S}{2}$, *if n is even.*



Reverse mean inequalities

Theorem

Let $C \in \mathcal{K}^n$ be Minkowski centered. Then

(i) $\mathrm{conv}(C \cup (-C)) \subset^{opt} s(C)(C \cap (-C))$,

(ii) $\mathrm{conv}(C \cup (-C)) \subset^{opt} \frac{2s(C)}{s(C)+1} \frac{C-C}{2}$,

(iii) $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset^{opt} \frac{2s(C)}{s(C)+1}(C \cap (-C))$,

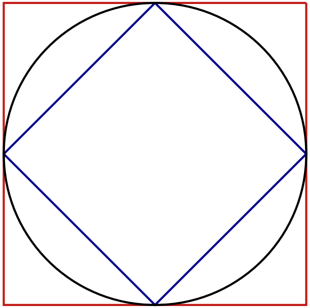
(iv) $\frac{C-C}{2} \subset^{opt} \frac{s(C)+1}{2}(C \cap (-C))$,

(v) $\mathrm{conv}(C \cup (-C)) \subset^{opt} \frac{s(C)+1}{2} \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$.

(vi) $\frac{C-C}{2} \subset \frac{s(C)+1}{2} \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$, *and for all* $s \in [1, n]$ $\exists$ *a Minkowski centered* $C \in \mathcal{K}^n$ *with* $s(C) = s$, *s.th. containment is optimal.*

Let $x \in \mathbb{R}^n$. Then

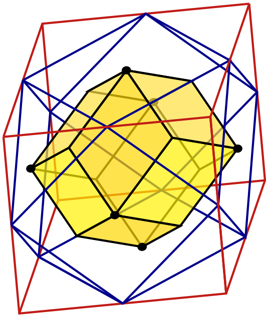
$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \quad \text{and} \quad \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty$$



$$\|x\|_{conv(C\cup(-C))} \leq \|x\|_{\frac{C-C}{2}} \leq \|x\|_{\left(\frac{C^\circ - C^\circ}{2}\right)^\circ} \leq \|x\|_{C\cap(-C)}$$

$$\|x\|_{C\cap(-C)} \leq \frac{s+1}{2}\|x\|_{\frac{C-C}{2}} \leq s\|x\|_{conv(C\cup(-C))}$$

$$\text{or } \|x\|_{C\cap(-C)} \leq \frac{2s}{s+1}\|x\|_{\left(\frac{C^\circ - C^\circ}{2}\right)^\circ} \leq s\|x\|_{conv(C\cup(-C))}$$



Theorem

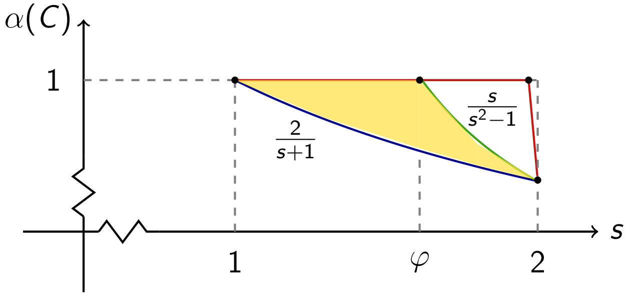
(i) $C \cap (-C) \subset^{opt} conv(C \cup (-C)) \iff$

(ii) $\left(\frac{1}{2}(C^\circ - C^\circ)\right)^\circ \subset^{opt} \frac{1}{2}(C - C) \iff$

(iii) $\exists p, -p \in bd(C)$, *parallel halfspaces* $H^{\leq}_{a,\rho}$, $H^{\leq}_{-a,\rho}$ *supporting* C *at* p, $-p$, *respectively.*

Let $C \in \mathcal{K}^n$, $\alpha(C) > 0$ s.th. $C \cap (-C) \subset^{opt} \alpha(C)\,\mathrm{conv}(C \cup (-C))$.
For $s \in [1, n]$ we define

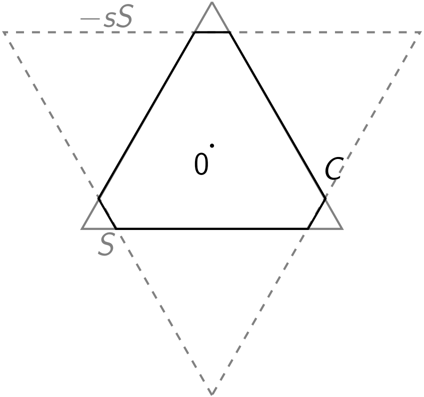
$$\alpha_{1\backslash 2}(s) := \inf \backslash \sup\{\alpha(C) : C \in \mathcal{K}^n \text{ Minkowski centered}, s(C) = s\}.$$



For $n = 2$: $\alpha_1(s) = \frac{2}{s+1}$. $\alpha_2(s) = 1$ for $s \leq \varphi := \frac{1+\sqrt{5}}{2}$;
$\alpha_2(s) \geq \frac{s}{s^2-1}$ for $s \geq \varphi$, $\alpha_2 \leq \frac{-26s^2+36s+34}{18s^2-24s-21}$ for $s > \varphi$.

For $C = S \cap (-sS)$ with S a regular Minkowski centered simplex holds

$$C \cap (-C) \subset^{opt} \frac{2}{s+1} \mathrm{conv}(C \cup (-C))$$



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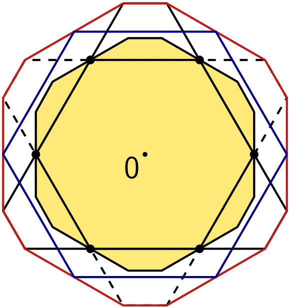
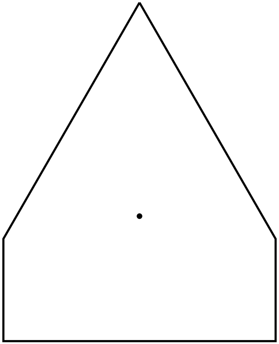


$$C \cap (-C) \subset^{opt} \max\{1, \frac{s}{s^2-1}\}\mathrm{conv}(C \cup (-C))$$



For $C = S \cap (-sS)$ holds

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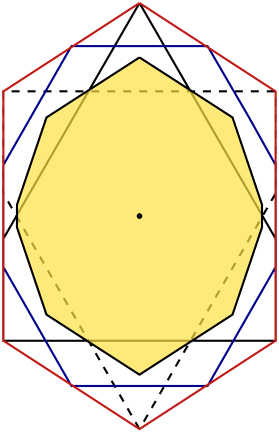
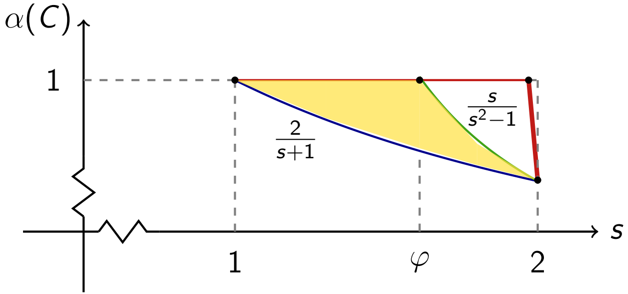


Let $C \in \mathcal{K}^n$, $\alpha(C) > 0$ s.th. $C \cap (-C) \subset^{opt} \alpha(C)\,\mathrm{conv}(C \cup (-C))$.
For $s \in [1, n]$ we define

$$\alpha_{1\setminus 2}(s) := \inf \setminus \sup\{\alpha(C) : C \in \mathcal{K}^n \text{ Minkowski centered}, s(C) = s\}.$$



For $n = 2$: $\alpha_1(s) = \frac{2}{s+1}$. $\alpha_2(s) = 1$ for $s \le \varphi := \frac{1+\sqrt{5}}{2}$;
$\alpha_2(s) \ge \frac{s}{s^2-1}$ for $s \ge \varphi$, $\alpha_2 \le \frac{-26s^2+36s+34}{18s^2-24s-21}$ for $s > \varphi$.

Theorem

Let $C \in \mathcal{K}^n$ be Minkowski centered with $s(C) = s$, n even. Then

(i) $C \cap (-C) \subset \psi \dfrac{n}{n+1} \mathrm{conv}(C \cup (-C))$, *if* $s \geq \gamma_2(n)$, *and*

(ii) $\left(\dfrac{C^\circ + (-C)^\circ}{2}\right)^\circ \subset \zeta \dfrac{n(n+2)}{(n+1)^2} \dfrac{C - C}{2}$, *if* $s \geq \gamma_3(n)$, *where*

$$\psi := \psi(n,s) := \frac{(n-s+1)(s+1)}{1-(n-s)(n+s(n+1))} - n,$$

$$\zeta := \zeta(n,s) = (n+1)\left(\left(1 + \frac{sn}{s+1}\right)\frac{1+n-s}{1-n(n-s)} - n\right),$$

$$\gamma_2 := \gamma_2(n) := \frac{n^4+n^3+2n^2+\sqrt{\delta_2}}{2(n^3+2n^2+3n+1)},$$

$$\delta_2 := \delta_2(n) := n^8 + 6n^7 + 17n^6 + 28n^5 + 28n^4 + 12n^3 - 4n^2 - 12n - 4.$$

Idea of the proof

For $C, K \in \mathcal{K}^n$ the **Banach-Mazur distance** between K and C is

$$d_{BM}(K, C) := \inf\{\rho \geq 1 : c^1 + K \subset L(C) \subset c^2 + \rho K,\ L \in \mathrm{GL}(n)\}.$$

Proposition (Schneider, 2009)

Let $S \in \mathcal{K}^n$ be an n-simplex and $C \in \mathcal{K}^n$ s.th. $s(C) = n - \varepsilon$, $\varepsilon \in (0, \frac{1}{n})$. Then

$$d_{BM}(C, S) \leq 1 + \frac{(n+1)\varepsilon}{1 - n\varepsilon}.$$

- For $s \in (n - \frac{1}{n}, n)$: $c^1 + S \subset L(C) \subset c^2 + \rho S$ with $\rho \leq \frac{n+1-s}{1-n(n-s)}$.

Theorem

Let $C \in \mathcal{K}^n$ be Minkowski centered, $\alpha \in \mathbb{R}$. Then

$$C \cap (-C) \subset^{opt} \alpha \cdot \mathrm{conv}(C \cup (-C)) \quad \textit{iff}$$

$$L(C) \cap L(-C) \subset^{opt} \alpha \cdot \mathrm{conv}(L(C) \cup L(-C)).$$

- Thus, $c^1 + S \subset C \subset c^2 + \rho S$ with $\rho \leq \frac{n+1-s}{1-n(n-s)}$.
- Show: $0 \in c^1 + S \subset C \subset c^2 + \rho S$.
- Define: $\bar{\mu}$ as minimal distance from 0 to the facets of $c^1 + S$.
- Show: $\frac{n+1}{s+1}(1 - s(\rho - 1)) =: \mu \leq \bar{\mu} \leq 1$.

- Combine:

$$
\begin{aligned}
C \cap (-C) &\subset (c^2 + \rho S) \cap (-c^2 - \rho S) \\
&\subset (\rho + n(\rho - \mu))(S \cap (-S)) \\
&\subset \frac{n}{n+1}(\rho + n(\rho - \mu))\mathrm{conv}(S \cup (-S)) \\
&\subset \frac{n}{n+1}\frac{(\rho + n(\rho - \mu))}{\mu}\mathrm{conv}((c^1 + S) \cup (-c^1 - S)) \\
&\subset \frac{n}{n+1}\frac{(\rho + n(\rho - \mu))}{\mu}\mathrm{conv}(C \cup (-C)).
\end{aligned}
$$

Let $C \in \mathcal{K}^n$, $\alpha(C) > 0$ s.th. $C \cap (-C) \subset^{opt} \alpha(C)\,\mathrm{conv}(C \cup (-C))$.
For $s \in [1, n]$ we define

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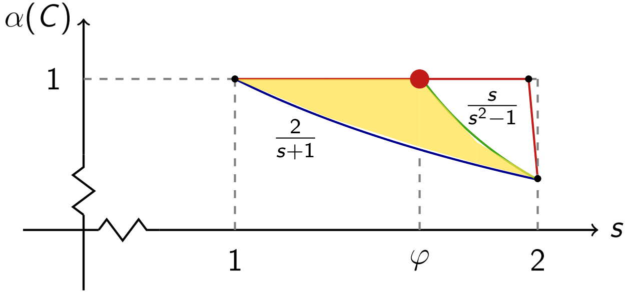

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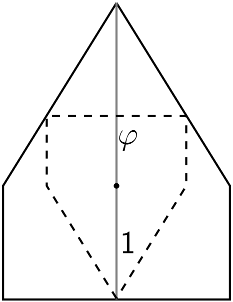
Golden House

Theorem

Let $C \in \mathcal{K}^2$ be Minkowski centered, $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset^{opt} \frac{C-C}{2}$, then $s(C) \leq \varphi$.
If $s(C) = \varphi$, $\exists$ linear transformation L s.th. $L(C) =$ golden house.



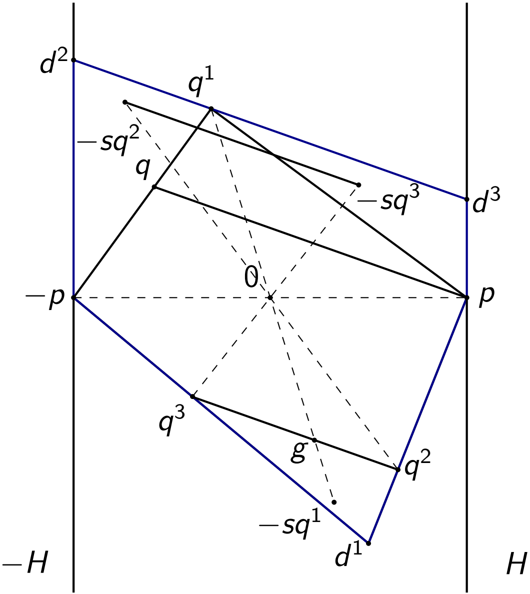
Idea of the proof

Proposition (Brandenberg, Koenig, 2013)

Let $K, C \in \mathcal{K}^n$ and $K \subset C$. The following are equivalent:

(i) $K \subset^{opt} C$.

(ii) *There exist $k \in \{2, \ldots, n+1\}$, $p^j \in K \cap \mathrm{bd}(C)$, $a^j \in N(C, p^j)$, $j = 1, \ldots, k$, such that $0 \in \mathrm{conv}(\{a^1, \ldots, a^k\})$.*

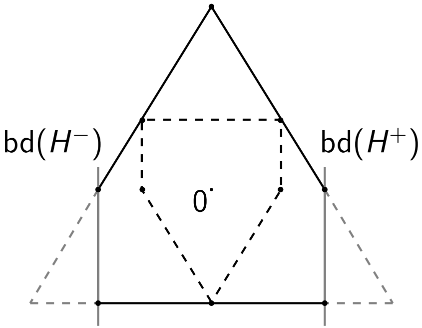


d^2
q^1
$-sq^2$
q
$-sq^3$
d^3
0
$-p$
p
q^3
g
q^2
$-sq^1$
d^1
$-H$
H

Generalized Golden House

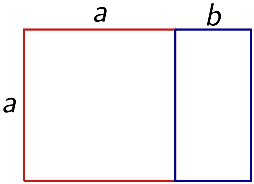
Let $n \geq 2$. Then for $C =$ generalized golden house holds

$$C \cap (-C) \subset^{opt} \mathrm{conv}(C \cup (-C)), \quad s(C) = \frac{1}{2}(n - 1 + \sqrt{(n-2)n+5})$$



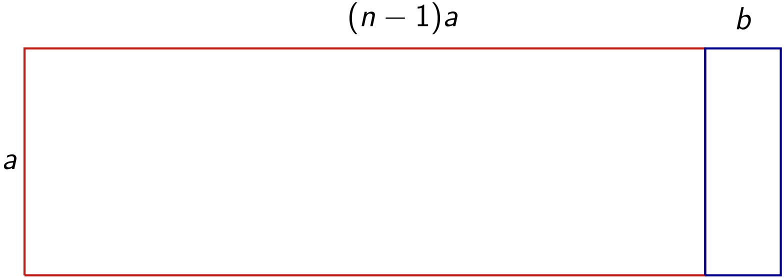
Golden Ratio

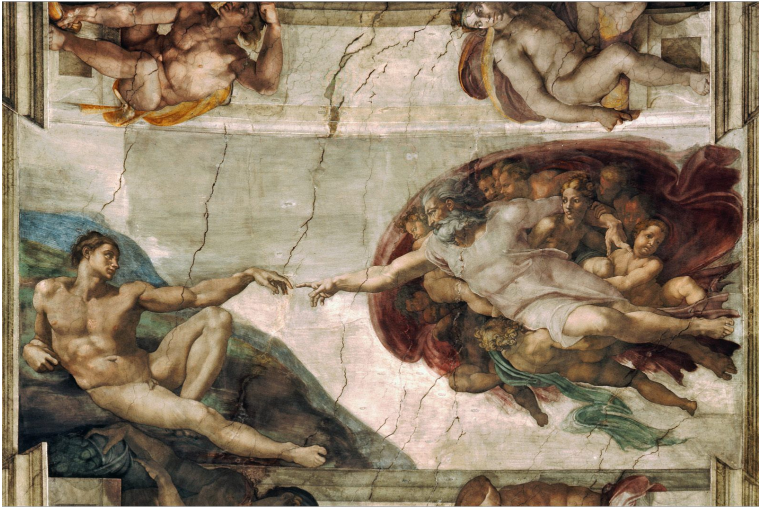
$$\frac{a+b}{a} = \frac{a}{b} =: \varphi \implies \varphi^2 - \varphi - 1 = 0 \implies \varphi = \frac{\sqrt{5}+1}{2}.$$



Generalized Golden Ratio

$$\frac{(n-1)a+b}{a} = \frac{a}{b} =: \bar{\varphi} \implies \bar{\varphi}^2 - (n-1)\bar{\varphi} - 1 = 0$$
$$\implies \bar{\varphi} = \frac{1}{2}\left(n - 1 + \sqrt{(n-2)n+5}\right).$$





Thank you for your attention!