Threshold for the expected measure of random polytopes

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Threshold for the volume

General question

Establish a threshold for the expected volume of a random polytope defined as the convex hull of independent random points with a given distribution.

The original, still vague, formulation: Let μ be a Borel probability on $\mathbb{R}^n$ such that $\mathrm{conv}(\mathrm{supp}(\mu))$ is a convex body K. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent random vectors X_i distributed according to μ. For each $N > n$ consider the random polytope

$$K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$$

and the normalized expectation of its volume

$$E_\mu(N) = \mathbb{E}_{\mu^N}\left(\frac{|K_N|}{|K|}\right).$$

We say that μ exhibits a threshold at $\varrho > 0$ if

$$E_\mu(N) \sim 0 \quad \text{when} \quad N \ll \exp(\varrho n)$$

and

$$E_\mu(N) \sim 1 \quad \text{when} \quad N \gg \exp(\varrho n).$$

Threshold for the volume

Dyer-Füredi-McDiarmid (1992)

Let μ_n be the uniform measure on $\{-1,1\}^n$. If $\varrho = \ln 2 - \frac{1}{2}$ (the same for every n) then for every $\epsilon \in (0,\varrho)$ we have that

$$\lim_{n\to\infty} \sup\left\{2^{-n}\mathbb{E}\,|K_N| : N \leqslant \exp((\varrho-\epsilon)n)\right\} = 0$$

and

$$\lim_{n\to\infty} \inf\left\{2^{-n}\mathbb{E}\,|K_N| : N \geqslant \exp((\varrho+\epsilon)n)\right\} = 1.$$

- Dyer, Füredi and McDiarmid also obtained a similar result for the case where μ_n is the uniform measure on $[-1,1]^n$. The value of the constant is now

$$\varrho = \ln(2\pi) - (\gamma + 1/2),$$

where γ is Euler's constant.

Threshold for the volume

- A very general guess is that the following might be true.

Statement

Let $\delta \in \left(0, \frac{1}{2}\right)$. There exists a sequence $\epsilon_n(\delta) \to 0$ as $n \to \infty$ and $n_0(\delta) \in \mathbb{N}$ such that if $n \geqslant n_0$ and K is a convex body in $\mathbb{R}^n$ then we may find a constant $\varrho = \varrho(K)$ such that

$$\sup \left\{ \mathbb{E}\left(\frac{|K_N|}{|K|}\right) : N \leqslant \exp((1 - \epsilon_n(\delta))\varrho n) \right\} \leqslant \delta$$

and

$$\inf \left\{ \mathbb{E}\left(\frac{|K_N|}{|K|}\right) : N \geqslant \exp((1 + \epsilon_n(\delta))\varrho n) \right\} \geqslant 1 - \delta,$$

where $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$ and X_i are independent random points from K.

- The first main question is to identify the constant $\varrho(K)$ and then to establish these estimates with a constant $\epsilon_n(\delta)$, ideally independent from K, which tends to 0 as $n \to \infty$.
- Apart from the results of Dyer, Füredi and McDiarmid, sharp results in the spirit of the statement above are known only in (very) special cases.

Related works

- M. E. Dyer, Z. Füredi and C. McDiarmid, *Volumes spanned by random points in the hypercube*, Random Structures Algorithms 3 (1992), 91–106.
- P. Pivovarov, *Volume thresholds for Gaussian and spherical random polytopes and their duals*, Studia Math. 183 (2007), no. 1, 15–34.
- D. Gatzouras and A. Giannopoulos, *Threshold for the volume spanned by random points with independent coordinates*, Israel J. Math. 169 (2009), 125–153.
- G. Bonnet, G. Chasapis, J. Grote, D. Temesvari and N. Turchi, *Threshold phenomena for high-dimensional random polytopes*, Commun. Contemp. Math. 21 (2019), no. 5, 1850038, 30 pp.
- G. Bonnet, Z. Kabluchko and N. Turchi, *Phase transition for the volume of high-dimensional random polytopes*, Random Structures Algorithms 58 (2021), no. 4, 648–663.
- A. Frieze, W. Pegden and T. Tkocz, *Random volumes in d-dimensional polytopes*, Discrete Anal. 2020, Paper No. 15, 17 pp.
- D. Chakraborti, T. Tkocz and B-H. Vritsiou, *A note on volume thresholds for random polytopes*, Geom. Dedicata 213 (2021), 423–431.

A more general problem

- Let μ be a log-concave probability measure μ on $\mathbb{R}^n$.
- This means that $\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^{\lambda}\mu(B)^{1-\lambda}$ for any compact subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$.
- It is known that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H in $\mathbb{R}^n$, then μ has a log-concave density f_{μ}.
- By the Brunn-Minkowski inequality, the uniform measure on a convex body K in $\mathbb{R}^n$ is log-concave.
- Let $X_1, X_2, \ldots$ be independent random points in $\mathbb{R}^n$ distributed according to μ and for any $N > n$ define the random polytope

$$K_N = \mathrm{conv}\{X_1, \ldots, X_N\}.$$

- Consider the expectation $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of the measure of K_N, where $\mu^N = \mu \times \cdots \times \mu$, N times.
- We may assume that μ is centered, i.e. the barycenter of μ is at the origin.
- If $\mu = \mu_K$, the uniform measure on a convex body K in $\mathbb{R}^n$, then this quantity is the same with the one we discussed before.

A more general problem

- Given $\delta \in (0,1)$ we say that μ satisfies a "δ-upper threshold" with constant ϱ_1 if

$$\sup\{\mathbb{E}_{\mu^N}[\mu(K_N)] : N \leqslant \exp(\varrho_1 n)\} \leqslant \delta \tag{1}$$

and that μ satisfies a "δ-lower threshold" with constant ϱ_2 if

$$\inf\{\mathbb{E}_{\mu^N}[\mu(K_N)] : N \geqslant \exp(\varrho_2 n)\} \geqslant 1-\delta. \tag{2}$$

- Then, we define

$$\varrho_1(\mu,\delta) = \sup\{\varrho_1 : (1) \text{ holds true}\} \quad \text{and} \quad \varrho_2(\mu,\delta) = \inf\{\varrho_2 : (2) \text{ holds true}\}.$$

- Our main goal is to obtain upper bounds for the difference

$$\pi(\mu,\delta) := \varrho_2(\mu,\delta) - \varrho_1(\mu,\delta)$$

for any fixed $\delta \in \left(0, \frac{1}{2}\right)$.

- One may also consider a sequence $\{\mu_n\}_{n=1}^{\infty}$ of log-concave probability measures μ_n on $\mathbb{R}^n$. Then, we say that $\{\mu_n\}_{n=1}^{\infty}$ exhibits a "sharp threshold" if there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$ of positive reals such that $\delta_n \to 0$ and $\pi(\mu_n,\delta_n) \to 0$ as $n \to \infty$.

Tukey's half-space depth

- Let μ be a probability measure on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ we denote by $\mathcal{H}(x)$ the set of all half-spaces H of $\mathbb{R}^n$ containing x.
- Tukey's half-space depth is the function

$$\varphi_\mu(x) = \inf\{\mu(H) : H \in \mathcal{H}(x)\}.$$

G.-Brazitikos-Pafis (2022)

If μ is a log-concave probability measure on $\mathbb{R}^n$ then

$$\exp(-c_1 n) \leqslant \int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x) \leqslant \exp\left(-c_2 n / L_\mu^2\right),$$

where $c_1, c_2 > 0$ are absolute constants and L_μ is the isotropic constant of μ.

- The right-hand side inequality answers a question from "math-overflow" and combined with the left-hand side inequality determines the expectation of φ since now it is known that $L_\mu \leqslant c\sqrt{\ln n}$.

The geometric lemmas

- For any convex body $B \subset \mathbb{R}^n$ we define

$$\varphi(B) = \inf_{x \in B} \varphi_\mu(x).$$

The first lemma

Let $B \subset \mathbb{R}^n$ be a convex body. For every $N > n$ we have

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant \mu(B) + N\,\varphi(B).$$

The second lemma

Let $B \subset \mathbb{R}^n$ be a convex body. For every $N > n$ we have

$$1 - \mathbb{P}(K_N \supseteq B) \leqslant 2\binom{N}{n}(1 - \varphi(B))^{N-n}.$$

Consequently,

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \geqslant \mu(B)\,\left(1 - 2\binom{N}{n}(1 - \varphi(B))^{N-n}\right).$$

The geometric lemmas

- The idea is to define an increasing family of convex bodies $\{B_t\}_{t>0}$, depending on μ, and determine the function
$$\varphi(t)=\varphi(B_t).$$
- Assume that we can also identify a value ϱ so that
$$\mu(B_{(1-\epsilon)\varrho n})\leqslant \delta/2 \quad \text{and} \quad \mu(B_{(1+\epsilon)\varrho n})\geqslant 1-\delta/2$$
for some small $\epsilon=\epsilon(\mu,\delta)$.
- If $N\leqslant \exp((1-2\epsilon)\varrho n)$ implies that $N\,\varphi(B_{(1-\epsilon)\varrho n})\leqslant \delta/2$ then the first lemma shows that
$$\sup\{\mathbb{E}_{\mu^N}(\mu(K_N)) : N\leqslant \exp((1-2\epsilon)\varrho n)\}\leqslant \mu(B_{(1-\epsilon)\varrho n})+N\,\varphi(B_{(1-\epsilon)\varrho n})\leqslant \delta,$$
and hence
$$\varrho_1(\mu,\delta)\geqslant (1-2\epsilon)\varrho n.$$

The geometric lemmas

- Recall that we assume that we can also identify a value ϱ so that

$$\mu(B_{(1-\epsilon)\varrho n}) \leqslant \delta/2 \quad \text{and} \quad \mu(B_{(1+\epsilon)\varrho n}) \geqslant 1-\delta/2$$

  for some small $\epsilon = \epsilon(\mu, \delta)$.

- If $N \geqslant \exp((1+2\epsilon)\varrho n)$ implies that

$$2\binom{N}{n}(1-\varphi(B_{(1+\epsilon)\varrho n}))^{N-n} \leqslant \delta/2,$$

  then the second lemma shows that

$$\inf\{\mathbb{E}_{\mu^N}(\mu(K_N)) : N \geqslant \exp((1+2\epsilon)\varrho n)\} \geqslant \mu(B_{(1+\epsilon)\varrho n}) \left(1 - 2\binom{N}{n}(1-\varphi(B_{(1+\epsilon)\varrho n}))^{N-n}\right) \geqslant 1-\delta,$$

  and hence

$$\varrho_2(\mu, \delta) \leqslant (1+2\epsilon)\varrho n.$$

- Then we have a threshold at ϱ with

$$\pi(\mu, \delta) := \varrho_2(\mu, \delta) - \varrho_1(\mu, \delta) \leqslant 4\epsilon\varrho.$$

The family $\{B_t\}_{t>0}$

- Let μ be a centered log-concave probability measure on $\mathbb{R}^n$ with density $f := f_\mu$.
- The logarithmic Laplace transform of μ on $\mathbb{R}^n$ is defined by

$$\Lambda_\mu(\xi) = \ln\Big(\int_{\mathbb{R}^n} e^{\langle \xi, z\rangle} f(z)dz\Big).$$

- It is easily checked that Λ_μ is convex and $\Lambda_\mu(0)=0$: Since $\mathrm{bar}(\mu)=0$, Jensen's inequality shows that

$$\Lambda_\mu(\xi) = \ln\Big(\int_{\mathbb{R}^n} e^{\langle \xi, z\rangle} f(z)dz\Big) \geqslant \int_{\mathbb{R}^n} \langle \xi, z\rangle f(z)dz = 0$$

  for all ξ.
- Therefore, Λ_μ is a non-negative function. One can check that the set $A(\mu)=\{\Lambda_\mu<\infty\}$ is open and Λ_μ is C^∞ and strictly convex on $A(\mu)$.

The family $\{B_t\}_{t>0}$

- We define
$$\Lambda_\mu^*(x) = \sup_{\xi\in\mathbb{R}^n} \{\langle x,\xi\rangle - \Lambda_\mu(\xi)\}.$$
- In other words, Λ_μ^* is the Legendre transform of Λ_μ: recall that given a convex function $g:\mathbb{R}^n\to(-\infty,\infty]$, the Legendre transform $\mathcal{L}(g)$ of g is defined by
$$\mathcal{L}(g)(x) := \sup_{\xi\in\mathbb{R}^n}\{\langle x,\xi\rangle - g(\xi)\}.$$
The function Λ_μ^* is called the Cramer transform of μ.

The family $\{B_t\}_{t>0}$

For every $t>0$ we also set
$$B_t(\mu) := \{x\in\mathbb{R}^n : \Lambda_\mu^*(x)\leqslant t\}.$$

Upper bound for $\varphi(B_t(\mu))$

Relation with half-space depth

Let μ be a Borel probability measure on $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ we have

$$\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x)).$$

In particular, for any $t > 0$ we have that

$$\varphi(B_t(\mu)) \leqslant \exp(-t).$$

- Let $x \in \mathbb{R}^n$. We start with the observation that for any $\xi \in \mathbb{R}^n$ the half-space $\{z : \langle z - x, \xi \rangle \geqslant 0\}$ is in $\mathcal{H}(x)$.
- Therefore,

$$\varphi_\mu(x) \leqslant \mu(\{z : \langle z, \xi \rangle \geqslant \langle x, \xi \rangle\}) \leqslant e^{-\langle x, \xi \rangle} \mathbb{E}_\mu\big(e^{\langle z, \xi \rangle}\big) = \exp\big(-[\langle x, \xi \rangle - \Lambda_\mu(\xi)]\big).$$

- Taking the infimum over all $\xi \in \mathbb{R}^n$ we see that $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$.

Lower bound for $\varphi(B_t(\mu))$

G.-Brazitikos-Pafis (2022)

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $t > 0$ we have that

$$\varphi(B_t(\mu_K)) = \inf\{\varphi_{\mu_K}(x) : x \in B_t(\mu_K)\} \geqslant \frac{1}{10}\exp(-t - 2\sqrt{n}),$$

where μ_K is Lebesgue measure on K.

- The first part of the argument works for any centered log-concave probability measure μ with density f on $\mathbb{R}^n$.
- Let $t > 0$. In order to give a lower bound for $\inf\{\varphi_\mu(x) : x \in B_t(\mu)\}$ it suffices to give a lower bound for $\mu(H)$, where H is any closed half-space whose bounding hyperplane supports $B_t(\mu)$. In that case,

$$\mu(H) = \mu(\{z : \langle z - x, \xi\rangle \geqslant 0\})$$

for some $x \in \partial(B_t(\mu))$, with $\xi = \nabla\Lambda_\mu^*(x)$, or equivalently $x = \nabla\Lambda_\mu(\xi)$.

Lower bound for $\varphi(B_t(\mu))$

- For every $\xi \in \mathbb{R}^n$ we define the probability measure μ_ξ with density

$$f_\xi(z) = e^{-\Lambda_\mu(\xi)+\langle \xi, z\rangle} f(z).$$

- The barycenter of μ_ξ is $x = \nabla\Lambda_\mu(\xi)$ and $\mathrm{Cov}(\mu_\xi) = \mathrm{Hess}\,(\Lambda_\mu)(\xi)$.
- We set

$$\sigma_\xi^2 = \mathbb{E}_{\mu_\xi}(\langle z-x,\xi\rangle^2) = \int_{\mathbb{R}^n} \langle z-x,\xi\rangle^2 d\mu_\xi(z).$$

- We compute that

$$\mu(\{z : \langle z-x,\xi\rangle \geqslant 0\}) \geqslant e^{-\Lambda_\mu^*(x)} \int_0^\infty \sigma_\xi e^{-\sigma_\xi t} \mu_\xi(\{z : 0 \leqslant \langle z-x,\xi\rangle \leqslant \sigma_\xi t\})\, dt$$

  and check that

$$\int_0^\infty \sigma_\xi e^{-\sigma_\xi t} \mu_\xi(\{z : 0 \leqslant \langle z-x,\xi\rangle \leqslant \sigma_\xi t\})\, dt \geqslant \frac{1}{10} e^{-2\sigma_\xi}.$$

Lower bound for $\varphi(B_t(\mu))$

Theorem (Nguyen, 2014)

Let ν be a log-concave probability measure on $\mathbb{R}^n$ with density $g = \exp(-p)$, where p is a convex function. Then,

$$\mathrm{Var}_\nu(p) \leqslant n.$$

- In the case where $\mu := \mu_K$, the density f_ξ of μ_ξ is proportional to $e^{\langle \xi, z\rangle}\mathbf{1}_K(z)$. Using the fact that

$$\mathbb{E}_{\mu_\xi}(\langle \xi, z\rangle) = \langle \nabla\Lambda_\mu(\xi), \xi\rangle = \langle x, \xi\rangle,$$

from the theorem of Nguyen we get that

$$\sigma_\xi^2 = \mathbb{E}_{\mu_\xi}(\langle z - x, \xi\rangle^2) = \mathrm{Var}_{\mu_\xi}(\langle \xi, z\rangle) \leqslant n.$$

- Then, for any bounding hyperplane H of $B_t(\mu)$ we have

$$\mu(H) \geqslant e^{-\Lambda_\mu^*(x)} \int_0^\infty \sigma_\xi e^{-\sigma_\xi t} \mu_\xi(\{z : 0 \leqslant \langle z - x, \xi\rangle \leqslant \sigma_\xi t\})\, dt$$
$$\geqslant \frac{1}{10} e^{-\Lambda_\mu^*(x) - 2\sigma_\xi} \geqslant \frac{1}{10}\exp(-t - 2\sqrt{n}).$$

Comparison of φ_{μ_K} and $\exp(-\Lambda^*_{\mu_K})$

We have actually proved a two-sided inequality.

Uniform measure on a convex body

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $x \in \mathrm{int}(K)$ we have that

$$\exp(-\Lambda^*_{\mu_K}(x)) \geqslant \varphi_{\mu_K}(x) \geqslant \frac{1}{10}\exp(-\Lambda^*_{\mu_K}(x) - 2\sqrt{n})$$

where μ_K is Lebesgue measure on K.

- Setting

$$\omega_{\mu_K}(x) = \ln\left(\frac{1}{\varphi_{\mu_K}(x)}\right)$$

  we get that, for every $x \in \mathrm{int}(K)$,

$$\omega_{\mu_K}(x) - 5\sqrt{n} \leqslant \Lambda^*_{\mu_K}(x) \leqslant \omega_{\mu_K}(x). \tag{3}$$

First basic question

Is it true that an analogue of (3) holds true for any centered log-concave probability measure μ on $\mathbb{R}^n$? This would allow us to apply the procedure that we describe to all log-concave probability measures.

The value of ϱ

- As we will see, for any centered log-concave probability measure μ on $\mathbb{R}^n$, the correct value of ϱ is
$$\varrho = \frac{1}{n}\mathbb{E}_\mu(\Lambda_\mu^*).$$
- Consider the parameter
$$\beta(\mu) = \frac{\mathrm{Var}_\mu(\Lambda_\mu^*)}{(\mathbb{E}_\mu(\Lambda_\mu^*))^2}.$$
- Roughly speaking, the plan is the following: provided that φ_μ is "almost constant" on $\partial(B_t(\mu))$ for all $t > 0$ and that $\beta(\mu) = o_n(1)$, we can establish a "sharp threshold" at ϱ for the expected measure of K_N with window
$$\pi(\mu, \delta) \leqslant c\sqrt{\beta(\mu)/\delta}\,\varrho.$$
- But, first of all, we would like to know for which centered log-concave probability measures μ on $\mathbb{R}^n$ the parameter $\beta(\mu)$ is well-defined. This is true if
$$\|\Lambda_\mu^*\|_{L^2(\mu)} = \left(\mathbb{E}_\mu\left((\Lambda_\mu^*)^2\right)\right)^{1/2} < \infty.$$
- A stronger sufficient condition is to require that Λ_μ^* has finite moments of all orders.

Moments of the Cramer transform

- Given $\kappa \in (0, 1/n]$ we say that a measure μ on $\mathbb{R}^n$ is κ-concave if $\mu((1-\lambda)A + \lambda B) \geqslant ((1-\lambda)\mu^\kappa(A) + \lambda\mu^\kappa(B))^{1/\kappa}$ for all compact subsets A, B of $\mathbb{R}^n$ with $\mu(A)\mu(B) > 0$ and all $\lambda \in (0,1)$.

G.-Brazitikos-Pafis (2022)

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0, 1/n]$ and let μ be a centered κ-concave probability measure with $\mathrm{supp}(\mu) = K$. Then,

$$\int_{\mathbb{R}^n} e^{\frac{\kappa \Lambda_\mu^*(x)}{2}} d\mu(x) < \infty.$$

Note that the uniform measure on a convex body is $1/n$-concave. The proof of the theorem is based on the next lemma.

Lemma (Chakraborti-Tkocz-Vritsiou)

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0, 1/n]$ and let μ be a centered κ-concave probability measure with $\mathrm{supp}(\mu) = K$. Then,

$$\varphi_\mu(x) \geqslant e^{-2}\kappa(1 - \|x\|_K)^{1/\kappa}$$

for every $x \in K$, where $\|x\|_K$ is the Minkowski functional of K.

Moments of the Cramer transform: Proof of the Theorem

- We know that $e^{\frac{\kappa\Lambda_\mu^*(x)}{2}} \leqslant \varphi_\mu(x)^{-\kappa/2}$ for all $x \in K$. From the lemma we also know that $\varphi_\mu(x) \geqslant e^{-2}\kappa(1-\|x\|_K)^{1/\kappa}$ for every $x \in K$. It follows that

$$\int_K e^{\frac{\kappa\Lambda_\mu^*(x)}{2}}\,d\mu(x) \leqslant (e^2/\kappa)^{\kappa/2}\int_K \frac{1}{(1-\|x\|_K)^{1/2}}\,d\mu(x).$$

- Let f denote the density of μ on K. We write

$$\begin{aligned}\int_K \frac{1}{(1-\|x\|_K)^{1/2}}\,d\mu(x) &= \int_{\mathbb{R}^n} \frac{f(x)}{(1-\|x\|_K)^{1/2}}\mathbf{1}_K(x)\,dx\\ &\leqslant \|f\|_\infty \int_K \frac{1}{(1-\|x\|_K)^{1/2}}\,dx\\ &= n|K|B(n,1/2)\|f\|_\infty\\ &\leqslant c\sqrt{n}\|f\|_\infty < +\infty.\end{aligned}$$

Moments of the Cramer transform: Summary

- We have proved the next theorem, which is in particular true for the uniform measure on any centered convex body.

Theorem

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0, 1/n]$ and let μ be a centered κ-concave probability measure with $\mathrm{supp}(\mu) = K$. Then, for all $p \geqslant 1$ we have that $\mathbb{E}_\mu\big((\Lambda_\mu^*(x))^p\big) < \infty$.

- Besides this, we can show that Λ_μ^* has finite moments of all orders in the following cases:
  - (i) If μ is a centered probability measure on $\mathbb{R}$ or a product of such measures.
  - (ii) If μ is a centered log-concave probability measure on $\mathbb{R}^n$ and there exists a function $g : [1, \infty) \to [1, \infty)$ with $\lim_{t\to\infty} g(t)/\ln(t+1) = +\infty$ such that $Z_t^+(\mu) \supseteq g(t)Z_2^+(\mu)$ for all $t \geqslant 2$, where $\{Z_t^+(\mu)\}_{t\geqslant 1}$ is the family of one-sided L_t-centroid bodies of μ.

Second basic question

Let μ be a log-concave probability measure μ on $\mathbb{R}^n$. Is it true that Λ_μ^* has finite moments of all orders? This would allow us to apply the procedure that we describe to all log-concave probability measures.

Moments of the Cramer transform: Estimates

- The method of proof of the Theorem gives in fact reasonable upper bounds for $\|\Lambda_\mu^*\|_{L^p(\mu)}$. If μ is a centered κ-concave probability measure with $\mathrm{supp}(\mu)=K$ then

$$\mathbb{E}_\mu(\Lambda_\mu^*) \leqslant \left(\mathbb{E}_\mu[(\Lambda_\mu^*)^2]\right)^{1/2} \leqslant \frac{c\ln n}{\kappa}\|f_\mu\|_\infty^{1/2}.$$

- In particular, if we assume that $\mu=\mu_K$ is the uniform measure on a centered convex body then we obtain a sharp two sided estimate for the most interesting case where $p=1$ or 2.

Theorem

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, $n\geqslant 2$. Then,

$$c_1 n/L_{\mu_K}^2 \leqslant \|\Lambda_{\mu_K}^*\|_{L^1(\mu_K)} \leqslant \|\Lambda_{\mu_K}^*\|_{L^2(\mu_K)} \leqslant c_2 n\ln n,$$

where L_{μ_K} is the isotropic constant of the uniform measure μ_K on K.

- The left-hand side inequality follows easily from Jensen inequality and the estimate $\int_{\mathbb{R}^n} e^{-\Lambda_\mu^*(x)}\,d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right)$ that we have proved in order to estimate the expectation of Tukey's half-space depth.
- Both the lower and the upper bound are of optimal order with respect to the dimension. This can be seen e.g. from the example of the uniform measure on the cube or the Euclidean ball.

Upper threshold: the final step

- The threshold is a consequence of Chebyshev's inequality. Recall that

$$\beta(\mu)=\frac{\mathrm{Var}_{\mu}(\Lambda_{\mu}^{*})}{(\mathbb{E}_{\mu}(\Lambda_{\mu}^{*}))^{2}}$$

  and $B_t(\mu)=\{x\in\mathbb{R}^n:\Lambda_{\mu}^{*}(x)\leqslant t\}$.
- Let $\varrho:=\frac{1}{n}\mathbb{E}_{\mu}(\Lambda_{\mu}^{*})$. Then, for all $\epsilon\in(0,1)$, from Chebyshev's inequality we have that

$$\begin{aligned}\mu(B_{(1-\epsilon)\varrho n}(\mu))&=\mu(\{\Lambda_{\mu}^{*}\leqslant(1-\epsilon)\varrho n\})\leqslant\mu(\{|\Lambda_{\mu}^{*}-\mathbb{E}_{\mu}(\Lambda_{\mu}^{*})|\geqslant\epsilon\mathbb{E}_{\mu}(\Lambda_{\mu}^{*})\})\\&\leqslant\frac{\mathrm{Var}_{\mu}(\Lambda_{\mu}^{*})}{\epsilon^{2}(\mathbb{E}_{\mu}(\Lambda_{\mu}^{*}))^{2}}=\frac{\beta(\mu)}{\epsilon^{2}}.\end{aligned}$$

- Choosing $\epsilon=\sqrt{2\beta(\mu)/\delta}$ we have that

$$\mu(B_{(1-\epsilon)\varrho n}(\mu))\leqslant\frac{\delta}{2}.$$

- As we have seen, this implies that

$$\varrho_1(\mu,\delta)\geqslant\left(1-\sqrt{8\beta(\mu)/\delta}\right)\varrho.$$

Lower threshold: the final step

- The argument is similar. For all $\epsilon \in (0,1)$, from Chebyshev's inequality we have that

$$1-\mu(B_{(1+\epsilon)\varrho n}(\mu)) \leqslant \mu(\{\Lambda_\mu^* \geqslant (1+\epsilon)\varrho n\}) \leqslant \mu(\{|\Lambda_\mu^* - \mathbb{E}_\mu(\Lambda_\mu^*)| \geqslant \epsilon \mathbb{E}_\mu(\Lambda_\mu^*)\})$$
$$\leqslant \frac{\mathrm{Var}_\mu(\Lambda_\mu^*)}{\epsilon^2(\mathbb{E}_\mu(\Lambda_\mu^*))^2} = \frac{\beta(\mu)}{\epsilon^2}.$$

- Choosing $\epsilon = \sqrt{2\beta(\mu)/\delta}$ we have that

$$1-\mu(B_{(1+\epsilon)\varrho n}(\mu)) \leqslant \frac{\delta}{2}.$$

- As we have seen, this implies that

$$\varrho_2(\mu,\delta) \leqslant (1+\sqrt{8\beta(\mu)/\delta})\varrho.$$

- In a few words, the final estimate is

$$\pi(\mu,\delta) \leqslant c_1\sqrt{\beta(\mu)/\delta}\,\varrho$$

and $\varrho \leqslant c_2 \ln n$.

The main questions

1. Let μ be a log-concave probability measure μ on $\mathbb{R}^n$ and define $\omega_\mu(x) = \ln\left(\frac{1}{\varphi_\mu(x)}\right)$. Is it true that
$$\|\omega_\mu - \Lambda_\mu^*\|_\infty = o(n)?$$
A stronger question is if
$$\|\omega_\mu - \Lambda_\mu^*\|_\infty = O(\sqrt{n}).$$
This is true for the uniform measure on any n-dimensional convex body.
2. Let μ be a log-concave probability measure μ on $\mathbb{R}^n$. Is it true that Λ_μ^* has finite moments of all orders? This is true for the uniform measure, and more generally for any κ-concave measure where $0 < \kappa \leqslant \frac{1}{n}$, on any n-dimensional convex body.
3. Estimate
$$\beta_n^* := \sup\{\beta(\mu_K) : K \text{ is a centered convex body of volume 1 in } \mathbb{R}^n\}$$
or, more generally,
$$\beta_n := \sup\{\beta(\mu) : \mu \text{ is a centered log-concave probability measure on } \mathbb{R}^n\}.$$
Is it true that $\lim_{n\to\infty} \beta_n = 0$?

Vertices with independent coordinates

- Let μ be an even Borel probability measure on the real line and consider a random variable X, on some probability space $(\Omega, \mathcal{F}, P)$, with distribution μ, i.e., $\mu(B) := P(X \in B)$ for all B in the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$.
- Let $X_1, \ldots, X_n$ be independent and identically distributed random variables, defined on the product space $(\Omega^n, \mathcal{F}^{\otimes n}, P^n)$, each with distribution μ. Set $\vec{X} = (X_1, \ldots, X_n)$ and, for a fixed N satisfying $N > n$, consider N independent copies $\vec{X}_1, \ldots, \vec{X}_N$ of $\vec{X}$.
- The distribution of $\vec{X}$ is $\mu_n := \mu \otimes \cdots \otimes \mu$ (n times) and the distribution of $(\vec{X}_1, \ldots, \vec{X}_N)$ is $\mu_n^N := \mu_n \otimes \cdots \otimes \mu_n$ (N times).
- Consider the random polytope

$$K_N := \mathrm{conv}\{\vec{X}_1, \ldots, \vec{X}_N\}$$

and for any $\delta \in \left(0, \frac{1}{2}\right)$ define $\pi(\mu_n, \delta)$ as before.

- We say that $\{\mu_n\}_{n=1}^{\infty}$ exhibits a sharp threshold if

$$\pi(\mu_n, \delta) := \varrho_2(\mu_n, \delta) - \varrho_1(\mu_n, \delta) \longrightarrow 0$$

as $n \to \infty$, for any fixed $\delta \in \left(0, \frac{1}{2}\right)$.

Vertices with independent coordinates

- Let
$$x^* = x^*(\mu) := \sup\{x \in \mathbb{R} \colon \mu([x,\infty)) > 0\}$$
be the right endpoint of the support of μ and set $I = (-x^*, x^*)$.
- Let
$$g(t) := \mathbb{E}\big(e^{tX}\big) := \int_{\mathbb{R}} e^{tx}\, d\mu(x), \qquad t \in \mathbb{R}$$
denote the moment generating function of X, and let $\Lambda_\mu(t) := \ln g(t)$ be its logarithmic moment generating function. By Hölder's inequality, Λ_μ is a convex function on $\mathbb{R}$.
- Consider the Legendre transform $\Lambda_\mu^* : I \to \mathbb{R}$ of Λ_μ; this is the function
$$\Lambda_\mu^*(x) := \sup\{tx - \Lambda_\mu(t) \colon t \in \mathbb{R}\}.$$
One can show that Λ_μ^* has finite moments of all orders.

Vertices with independent coordinates

- We say that μ is *admissible* if it is non-degenerate, i.e. $\mathrm{Var}(X) > 0$, and satisfies the following conditions:
  (i) There exists $r > 0$ such that $\mathbb{E}(e^{tX}) < \infty$ for all $t \in (-r, r)$; in particular, X has finite moments of all orders.
  (ii) If $x^* < \infty$ then $P(X = x^*) = 0$.
  (iii) One of the following holds: (1) $x^* < +\infty$ or (2) $x^* = +\infty$ and $\{\Lambda_\mu < \infty\} = \mathbb{R}$, or (3) $x^* = +\infty$, $\{\Lambda_\mu < \infty\}$ is bounded and μ is log-concave.
- Finally, we say that μ satisfies *the* Λ^**-condition* if

$$\lim_{x \uparrow x^*} \frac{-\ln \mu([x, \infty))}{\Lambda_\mu^*(x)} = 1.$$

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Let μ be an admissible even probability measure on $\mathbb{R}$ that satisfies the Λ^*-condition. Then, for any $\delta \in \left(0, \frac{1}{2}\right)$ we have that

$$\lim_{n \to \infty} \pi(\mu_n, \delta) = 0.$$

Vertices with independent coordinates

- A variant of the question, in which $\mu_n(K_N)$ is replaced by the volume of K_N, had been studied in the case where μ is compactly supported. Define

$$\kappa = \kappa(\mu) := \frac{1}{2x^*}\int_{-x^*}^{x^*} \Lambda_\mu^*(x)dx.$$

The next result generalized the work of Dyer, Füredi and McDiarmid.

Gatzouras-G.-2009

Let μ be an even, compactly supported, Borel probability measure on the real line and assume that $0 < \kappa(\mu) < \infty$. For every $\epsilon > 0$,

$$\lim_{n\to\infty} \sup \left\{(2x^*)^{-n}\mathbb{E}(|K_N|) \colon N \leqslant \exp((\kappa-\epsilon)n)\right\} = 0$$

and if μ satisfies the Λ^*-condition then we also have that

$$\lim_{n\to\infty} \inf \left\{(2x^*)^{-n}\mathbb{E}(|K_N|) \colon N \geqslant \exp((\kappa+\epsilon)n)\right\} = 1.$$

Vertices with independent coordinates

- The result can be applied to the case of the product p-measure $\nu_p^n := \nu_p^{\otimes n}$. For any $p \geqslant 1$ we denote by ν_p the probability distribution on $\mathbb{R}$ with density $(2\gamma_p)^{-1}\exp(-|x|^p)$, where $\gamma_p = \Gamma(1+1/p)$.

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For any $p \geqslant 1$ we have that

$$\lim_{x\to\infty} \frac{-\ln(\nu_p[x,\infty))}{\Lambda^*_{\nu_p}(x)} = 1.$$

- Note that the measure ν_p is admissible for all $1 \leqslant p < \infty$; it satisfies condition (iii-3) if $p = 1$ and condition (iii-2) for all $1 < p < \infty$.
- Therefore, if K_N is the convex hull of $N > n$ independent random vectors $\vec{X}_1, \dots, \vec{X}_N$ with distribution ν_p^n then the expected measure $\mathbb{E}_{(\nu_p^n)^N}(\nu_p^n(K_N))$ exhibits a sharp threshold at $N = \exp((1\pm\epsilon)\mathbb{E}_{\nu_p}(\Lambda^*_{\nu_p})n)$; for any $\delta \in \left(0, \frac{1}{2}\right)$ we have that $\lim_{n\to\infty} \pi(\nu_p^n, \delta) = 0$.

A final example: Euclidean ball

- Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Recall the definition of $\omega_{\mu_K} = \ln(1/\varphi_{\mu_K})$ and consider the parameter
$$\tau(\mu_K) = \mathrm{Var}_{\mu_K}(\omega_{\mu_K})/(\mathbb{E}_{\mu_K}(\omega_{\mu_K}))^2.$$
- A useful observation is that we can estimate $\beta(\mu_K)$ if we can compute $\tau(\mu_K)$. We have that
$$\beta(\mu_K) = \left(\tau(\mu_K) + O(L_{\mu_K}^2/\sqrt{n})\right)\left(1 + O(L_{\mu_K}^2/\sqrt{n})\right).$$
Doing this for the Euclidean ball, we get:

Threshold for the ball

Let D_n be the centered Euclidean ball of volume 1 in $\mathbb{R}^n$. Then, the sequence $\mu_n := \mu_{D_n}$ exhibits a sharp threshold with

$$\pi(\mu_n, \delta) \leqslant \frac{c}{\sqrt{\delta}}\sqrt{\frac{\ln n}{n}}$$

and e.g. if n is even then we have that

$$\varrho_n = \frac{1}{n}\mathbb{E}_{\mu_n}(\Lambda_{\mu_n}^*) = \frac{(n+1)}{2n}H_{\frac{n}{2}} + O(1/\sqrt{n})$$

as $n \to \infty$, where $H_m = \sum_{k=1}^m \frac{1}{k}$.