Submodularity and Fractional Subadditivity in Geometric Functional Analysis & Information Theory

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Properties of Set Functions

Let $[n] = \{1, 2, \ldots, n\}$ and $v : 2^{[n]} \to \mathbb{R}$ (or $\overline{\mathbb{R}}$).

We say that v is <u>submodular</u> if

$$v(s \cup t) + v(s \cap t) \leq v(s) + v(t)$$

$\forall\ s, t \subset [n]$.

If $-v$ is submodular, we say v is <u>supermodular</u>.

<u>E.g:</u> 1) Cardinality $v(s) = |s|$ is submodular and supermodular.

2) Rank function of a matroid is submodular.

Examples from Geom. Func. Analysis & IT

1) For a discrete r.v. X, let $H(X) = \sum_{a \in \text{Support}(X)} - p_X(a) \log p_X(a)$.

If $X_1, \dots, X_n$ are arbitrary discrete r.v's on the same probability space,

$\nu(s) = H(X_s)$ is submodular, where $X_s = \{X_i : i \in s\}$.

Why?. $H(X_1, X_2) + H(X_1, X_3) - H(X_1, X_2, X_3) - H(X_1)$

$$= \mathbb{E}\left[\log \frac{1}{p_{X_1,X_2}(x_1,x_2)} + \dots\right.$$

$= \underbrace{\text{average of relative entropies}}$ ≥ 0.

2) For compact, convex sets $A_1, \ldots, A_n \subset \mathbb{R}^d$,

$$v(S) = \mathrm{Vol}_d\Big(\sum_{i \in S} A_i\Big) \quad \text{is} \quad \text{supermodular}.$$

Explicitly in [FMMZ '18], perhaps folklore?

Why? Suffices to prove that

$$\mathrm{Vol}_d(B_1 + B_2 + B_3) + \mathrm{Vol}_d(B_1) \geq \mathrm{Vol}_d(B_1 + B_2) + \mathrm{Vol}_d(B_1 + B_3).$$

Recall the mixed volume formula:

$$\circledast = \mathrm{Vol}_d(\underbrace{B_1 + B_2} + B_3) - \mathrm{Vol}_d(\underbrace{B_1 + B_2}) = \sum_{k=1}^{d} \binom{d}{k} V\big((B_1 + B_2)[d-k], B_3[k]\big)$$

Can assume $0 \in B_2$ by translation-invariance, so $B_1 \subset B_1 + B_2$.
Monotonicity property implies

$$\circledast \geq \sum_{k=1}^{d} \binom{d}{k} V\big(B_1[d-k], B_3[k]\big) = \mathrm{Vol}_d(B_1 + B_3) - \mathrm{Vol}_d(B_1).$$

Fractional subadditivity

We say that $\nu : 2^{[n]} \to \mathbb{R}$ is **fractionally subadditive**
if $$\nu([n]) \leq \sum_{S \subset [n]} \alpha(S)\, \nu(S) \qquad \forall \text{ fractional partition } \alpha.$$

Rmks: 1) $\alpha : 2^{[n]} \to [0,\infty)$ is a fractional partition if

$$\sum_{S \subset [n]} \alpha(S)\, \mathbb{1}_S = 1, \quad \text{i.e.,} \quad \sum_{\substack{S \subset [n] \\ S \ni i}} \alpha_S = 1 \quad \forall i \in [n].$$

2) $\mathrm{Supp}(\alpha) = \{ S \subset [n] : \alpha(S) > 0 \}$ is a hypergraph.

Generalizes partitions.

If $\alpha(S) \in \{0,1\}$, $\displaystyle\sum_{\substack{S \subset [n] \\ S \ni i}} \alpha_S = 1$

SM and FSA

Thm: If $v : 2^{[n]} \to \mathbb{R}$ is submodular and $v(\phi) = 0$, then v is fractionally subadditive.

Rmks: 1) Consider s, t disjoint in
$$v(s \cup t) + v(s \cap t) \leq v(s) + v(t).$$

2) History:
- Implicit in cooperative game theory 1950's (Shaply's core for convex games + Bondareva – Shaply theorem)
- Direct proof by Moulin-Ollagnier-Pinchon '82, c.f., M.-Tetali '10.
- Useful, e.g., Shearer's inequality.

The Analogies between Convex Geometry & Information Theory

Category of sets A	Category of functions f	Category of prob. measures μ
$A \subset \mathbb{R}^d$	$\mathbb{1}_A$	$\mathrm{Unif}(A)$
$\mathrm{Vol}_d(A)$	$\int_{\mathbb{R}^d} f$	$h(\mu) = -\int f \log f$ if $f = \frac{d\mu}{dx}$.
Convex sets	Log-concave or s-concave functions	Log-concave or s-concave measures

Examples of analogies

BMI $\quad A, B \subset \mathbb{R}^d$ compact

$$\mathrm{Vol}_d(A+B)^{1/d} \geq \mathrm{Vol}_d(A)^{1/d} + \mathrm{Vol}_d(B)^{1/d}$$

EPI $\quad X \in \mathbb{R}^d$

If $N(X) = \exp\left\{\frac{2h(X)}{d}\right\}$,

$$N(X+Y) \geq N(X) + N(Y)$$

if X, Y independent.

$$N(f * g) \geq N(f) + N(g)$$

Entropy of Convolutions : EPI's

Entropy power inequality :

If X, Y are independent, $N(X+Y) \geq N(X) + N(Y)$.

Rmks :

1) If X_1, X_2 i.i.d., using $N(aX) = a^2 N(X)$,

$$N\left(\frac{X_1+X_2}{\sqrt{2}}\right) \geq N(X_1)$$

$\Rightarrow$ in the CLT, $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$, where X_i are i.i.d

$N(S_{2^n})$ is $\uparrow$

Artstein-Ball-Barthe-Naor '04:

$$N\left(\sum_{i=1}^{n} X_i\right) \geq \frac{1}{n-1} \sum_{j=1}^{n} N\left(\sum_{i \neq j} X_i\right)$$

Barron-M. '07, Ghassemi-M. '18: If X_i are independent r.v's in $\mathbb{R}^d$,

$$\nu_{EP}(s) = N\left(\sum_{i \in s} X_i\right)$$

is fractionally superadditive.

$$\sum_{\substack{s \subset [n] \\ s \ni i}} \alpha_s = 1 \quad \forall i \; :$$

E.g.: $\alpha(s) = \begin{cases} 0, & \text{if } |s| \neq n-1 \\ \frac{1}{n-1}, & \text{if } |s| = n-1. \end{cases}$

Volumes of Minkowski sums

Brunn-Minkowski inequality (BMI) tells us that if $A_1, \dots, A_n$ are compacts in $\mathbb{R}^d$,

$$\nu_{BM}(s) = \mathrm{Vol}_d\Big(\sum_{i\in s} A_i\Big)^{1/d}$$

is superadditive.

Conjecture [Bobkov-Madiman-Wang '11]:

ν_{BM} is fractionally superadditive.

Rmk: ν_{BM} cannot be supermodular: [Fradelizi-Madiman-Marsiglietti-Zvavitch '18]

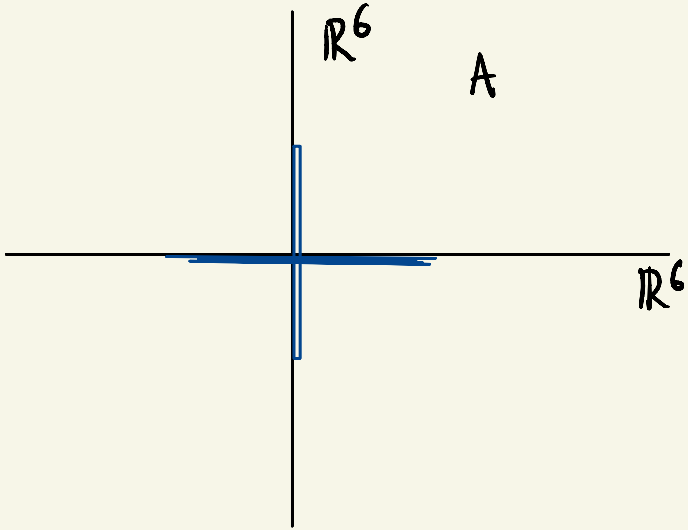
$$A = \{0,1\}, \quad B = C = [0,1]$$

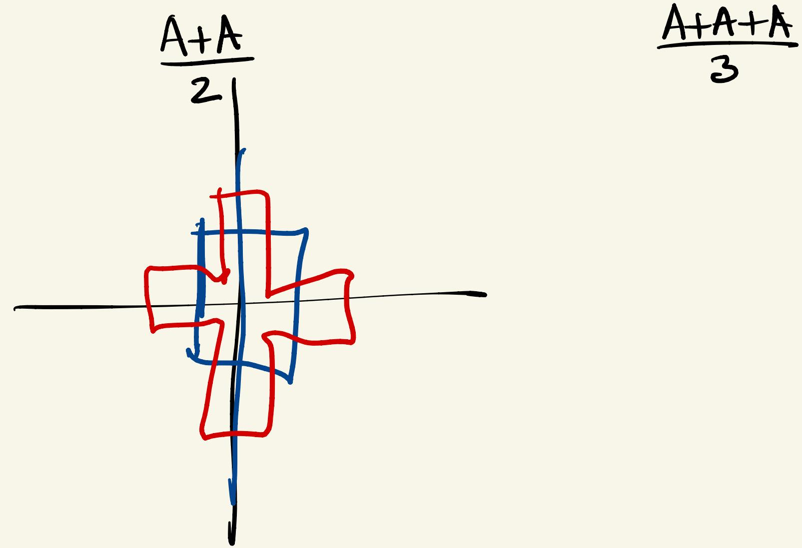
$$|A+B+C| + |A| = 3 < 4 = |A+B| + |A+C|.$$

Negative result on conjecture

Thm: [FMMZ'18]

For $d \geqslant 12$, ν_{BM} is not fractionally superadditive.





Rmk: [Fradelizi-Langi-Zvavitch'20] rule it out for $d \geqslant 7$
+ other interesting results.

Positive result on Conjecture

Thm: [Barthe - Madiman '21]

ν_{BM} is fractionally superadditive for $d=1$.

More generally, $\nu_{Vol}(S) = Vol_d\left(\sum_{i \in S} A_i\right)$ is fractionally superadditive for $A_1, \ldots, A_n \subset \mathbb{R}^d$, for any d.

Rmk: Uses total ordering of $\mathbb{R}$, inspired by Gyarmati-Matolcsi-Ruzsa '09.

$$\mathbb{1}_{\{1,2,3\}} = \frac{1}{2}\left[\mathbb{1}_{\{1,2\}} + \mathbb{1}_{\{2,3\}} + \mathbb{1}_{\{3,1\}}\right] \qquad n=3$$

$A_1, A_2, A_3 \subset \mathbb{R}$ compact with $\min A_i = 0$, $a_i = \max A_i$.

Since $0 \in A_i \ \forall i$,

$$(A_1 + A_2) \cup {}_{a_1+a_2<}(a_1 + A_2 + A_3) \subset A_1 + A_2 + A_3$$

$$(A_2 + A_3)_{\leq a_2} \cup (A_1 + a_2 + A_3) \subset A_1 + A_2 + A_3.$$

<u>Not^n:</u> ${}_{t<}S = S \cap (t, \infty)$, $S_{\leq t} = S \cap (-\infty, t]$.

$$2\,|A_1 + A_2 + A_3| \geq |A_1 + A_2| + |{}_{a_1+a_2<}(a_1 + A_2 + A_3)| + |(A_2 + A_3)_{\leq a_2}|$$
$$+ |A_1 + a_2 + A_3|$$
$$= |A_1 + A_2| + |A_2 + A_3| + |A_1 + A_3|.$$

Supermodularity Conjecture

Conj: [FMMZ'18]

For any convex body A and compacts B, C in $\mathbb{R}^d$,

$$\mathrm{Vol}_d(A+B+C) + \mathrm{Vol}_d(A) \geq \mathrm{Vol}_d(A+B) + \mathrm{Vol}_d(A+C).$$

Rmks:

1) $|A+B+C| + |\mathrm{conv}(A)| \geq |A+B| + |A+C|$

true for compacts $A, B, C \subset \mathbb{R}$ [FMMZ'18]

2) Conj. is true if B is a zonoid. [FMZ'22].

Entropy of Convolutions : Submodularity

Th^m: If $X_1, \dots, X_n$ are independent r.v.'s in a LCA group,

$$\nu_E(S) = h\Big(\sum_{i \in S} X_i\Big)$$

[Madiman '08, ...]

is submodular.

Rmks:

1) Suffices to show $h(X_1+X_2+X_3) + h(X_1)$
$$\leq h(X_1+X_2) + h(X_1+X_3).$$

2) Note that subadditivity is FALSE; $\nu(\phi) = -\infty$.

3) Contrast with superadditivity of $e^{2h(\sum_{i \in S} X_i)}$ on $\mathbb{R}$.

4) Very useful, e.g., reverse EPI for log-concave measures

[Bobkov-Madiman '11-'13]

Log-submodularity of Volume?

No hope without convexity assumptions.

Motivations:

Q: If A, B, C convex bodies in $\mathbb{R}^d$,

$$\mathrm{Vol}_d(A+B+C)\,\mathrm{Vol}_d(A) \leq \mathrm{Vol}_d(A+B)\,\mathrm{Vol}_d(A+C)?$$

Thm: [Bobkov-M'12]

$$\mathrm{LHS} \leq 3^d \cdot \mathrm{RHS}.$$

$h(X) \leq \log \mathrm{Vol}_d(\mathrm{supp}(X))$, with eq. for Uniform.
$h(X) \geq \log \mathrm{Vol}_d - c_{d,s}$, if X is s-concave, $s>0$.

[Nayar–Tkocz, Fradelizi–M–Zvavitch '22]

If we set

$$c_d = \sup_{\substack{A,B,C \\ \text{convex}}} \frac{\mathrm{Vol}_d(A+B+C)\,\mathrm{Vol}_d(A)}{\mathrm{Vol}_d(A+B)\,\mathrm{Vol}_d(A+C)},$$

$c_d > 1$, in fact, $c_d \geq \left(\frac{4}{3} + o(1)\right)^d$.

$c_d \leq \left(\frac{1+\sqrt{5}}{2}\right)^d$.

$c_2 = 1$, $c_3 = \frac{4}{3}$, ...?

Conj: $\gamma_{\log \mathrm{Vol}}(S) = \log \mathrm{Vol}_d \left(\sum_{i \in S} A_i \right)$

is submodular if $A_1, \ldots, A_n$ are zonoids.

Rmk: True in dim 3 [FMMZ '22].