MIDTERM, PROBABILITY I, FALL 2016

Each problem is worse 25 points. To get the full score (100 points) please select 4 out of 6 problems of your choice and solve them. Please indicate which problems you select at the first page of the exam; only those problems shall be graded. This scheme is used for qualifying exams, and the problems at the qualifying exam shall be similar in difficulty.

1. Let $p \geq 1$. Show that if $\mathbb{E}|X - X_n|^p \to 0$, and X_n converges to Y almost surely, then $X = Y$ almost surely.

2. Consider the sample space $\Omega = \{0, 1, 2\} \times \{0, 1, 2\}$ with uniform probability measure on Ω. Using the notation $\omega = (\omega_1, \omega_2)$, consider random variables $X(\omega) = \omega_1$ and $Y(\omega) = \omega_2$. Define $A = X$, $B = (X + Y) \mod 3$ and $C = (X + 2Y) \mod 3$. Show that A, B, C are pairwise independent, but not jointly independent.

3. For a $p > 0$, let $X = (X_1, ..., X_n)$ be a random vector distributed according to the density $f(x) = 1_Q \cdot (p + 1)^n \cdot \prod_{i=1}^n x_i^p$, where $Q = [0, 1]^n = \{x \in \mathbb{R}^n : x_i \in [0, 1] \, \forall i = 1, ..., n\}$. Assume that for every $\delta > 0$ there exists a positive integer N so that for all $n \geq N$,

$$P(|X| \in \sqrt{n}[\frac{\sqrt{3}}{2} - \delta, \frac{\sqrt{3}}{2} + \delta]) \geq 0.1.$$

Find p.

4. Assume that $P(\limsup_{n\to\infty} A_n) = 1$ and $P(\liminf_{n\to\infty} B_n) = 1$. Prove that

$$P(\limsup_{n\to\infty}(A_n \cap B_n)) = 1.$$

5. Suppose $X_n, n = \{1, 2, ...\}$ and X are random variables with bounded first moments. Assume that $X_n \geq 0$ almost surely, $\mathbb{E}X_n = 1$ and $\mathbb{E}(X_n \log X_n) \leq 1$. Assume that for every bounded random variable Y, $\mathbb{E}(X_n Y) \to_{n\to\infty} \mathbb{E}(XY)$. Show that:

- $X \geq 0$ almost surely;
- $\mathbb{E}X = 1$;
- $\mathbb{E}(X \log X) \leq 1$.

6. Let $X_1, X_2, ...$ be independent random variables with $\mathbb{E}X_i = \mu_i < \infty$ and $Var(X_i) = \sigma_i^2 < \infty$. Let $S_n = X_1 + ... + X_n$. Show that

$$P\left(\max_{1\leq k\leq n}\left|S_k - \sum_{i=1}^{k} \mu_i\right| \geq t\right) \leq \frac{1}{t^2}\sum_{i=1}^{n} \sigma_i^2.$$