Isobarycentric inequalities

Joint with Shani Gilboa and Boaz Slomka

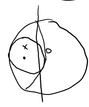
Def: Barycenter of $A \subseteq \mathbb{R}^n$ $b(A) = \frac{1}{\mathrm{vol}(A)} \int_A x\,dx$

$b(A) \in \mathrm{int}(\mathrm{conv}(A))$

$K \quad \sup\{|A| : A \subseteq K,\ b(A) = x\}$

$K \cap (2x - K)$

Example:



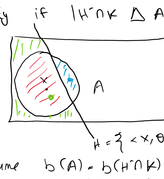
$x = -\alpha e_1$

$H = \{x \mid x_1 = x\}$

$b(H^- \cap B) = x.$

Lemma: Let H be a hyperplane. For every $A \subset K$ with $b(A) = b(H^- \cap K)$ we have $|A| \le |H^- \cap K|$ with equality if $|H^- \cap K \,\triangle\, A| = 0$

Pf:



Assume $b(A) = b(H^- \cap K) = 0$

$$|H^- \cap A| = \frac{\langle b(H^- \cap A^c \cap K), \theta\rangle}{\langle b(H^- \cap A), \theta\rangle} |H^- \cap A^c \cap K| < |H^- \cap A^c \cap K|$$

Let μ be a finite absolutely continuous Borel measure on $\mathbb{R}^n$ $b_\mu(A) = \frac{1}{\mu(A)} \int_A x\,d\mu(x)$

Thm: (Gilboa, H., Slomka) For every $b(K) \neq x \in \mathrm{int}(\mathrm{conv}(\mathrm{supp}(\mu)))$ there exists a half-space H^- which maximize μ among all subsets whose barycenter is x. Moreover, if $H \cap \mathrm{int}(\mathrm{supp}(\mu)) \neq \emptyset$ then H is unique.

Lemma: If $H_1 \neq H_2$ s.t. $H_i \cap \mathrm{int}(K) \neq \emptyset$, then $b(H_1^- \cap K) \neq b(H_2^- \cap K)$

Pf: If $b(K \cap H_1^-) = b(K \cap H_2^-) \Rightarrow |K \cap H_1^-| = |K \cap H_2^-|$

$\Rightarrow |K \cap H_1^- \,\triangle\, K \cap H_2^-| = 0$ a contradiction.

Lemma: For every $x \in \mathrm{int}(\mathrm{conv}(K))$ there exist a half-space H^- s.t. $b(K \cap H^-) = x$. [K bounded and connected]

Pf: Assume $x = 0$.

$H^-_{\theta, c} = \{x, \langle x, \theta\rangle < c\}$, $\theta \in S^{n-1}$ $\quad 0 \le c \le h_{\mathrm{conv}(K)}(\theta) = \|\theta\|_{K^\circ}$

$y \in K^\circ$, $y \neq 0$ $\quad \tilde{H}_y := \{x : \langle x, \frac{y}{\|y\|}\rangle = \|y\|_{K^\circ} \|\frac{y}{\|y\|}\|_{K^\circ}\}$

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$y \in \partial K^\circ \quad b(K \cap \tilde{H}_y^-) = b(K)$

$y \in K^\circ$, $\|y\| \to 0$ $\quad b(K \cap \tilde{H}_y^-)$ have a diff: $H_y := \{x : \langle x, \frac{y}{\|y\|}\rangle = (1 - \|y\|_{K^\circ}) \|\frac{y}{\|y\|}\|_{K^\circ}\}$ $\quad$ $\uparrow$ int

$$f(y) := \begin{cases} b(K \cap H_y^-) & y \neq 0 \\ b(K) & y = 0 \end{cases} \qquad f : K^\circ \to f(K^\circ) \subseteq K$$

Fact: f is continuous.

By previous lemma f is a bijection and hence a homeomorphism. $f(\partial K^\circ) = \partial f(K^\circ)$

$f(\partial K^\circ)$ is the set of $b(K \cap H^-)$ for $0 \in H$, we get $b(K) \in \mathrm{int}\, f(K^\circ)$

Want to show $0 \in f(K^\circ)$, it is enough to prove $[0, x(K)] \cap \partial f(K^\circ) = \emptyset$

Assume by contradiction $\lambda b(K) \in \partial f(K^\circ)$

$0 \le \lambda \le 1$

$\lambda b(K) = b(K \cap H_y^-)$

$$\lambda b(K) |K \cap H_y^-| + b(K \cap H_y^-) |K \cap H_y^+| = b(K) |K|$$

$$b(K \cap H_y^-) |K \cap H_y^+| = b(K) [(1-\lambda) |K \cap H_y^-| + |K \cap H_y^+|] = \alpha b(K)$$

$\alpha > 0$

a contradiction.

Generalization for measures:

- If $\mathrm{supp}(\mu)$ is bounded and connected.
- For the general case one can add $E\mu_0$, where μ_0 is the Gaussian measure and we restrict the measure to large ball

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The Log-Minkowski inequality

Conj: (Böröczky, Lutwak, Yang, Zhang)

$K, L \subseteq \mathbb{R}^n$ o-sym.

$$\int_{S^{n-1}} \log\left(\frac{h_K(u)}{h_L(u)}\right) dV_L(u) \le \frac{\mathrm{vol}(L)}{n} \log \frac{\mathrm{vol}(K)}{\mathrm{vol}(L)}$$

Question: (Henk-Pollehn) Does log-Minkowski hold when L is parallelotope and $b(K) = 0$?

After some affine transformations

$\forall\, 0 \neq x \in [-1, 1]^n$, $K \subseteq [-1, 1]^n$ s.t. $b(K) = x$,

$$\mathrm{Vol}(K) < 2^n \prod_{i=1}^n \sqrt{1 - x_i^2} \quad (*)$$

Henk-Pollehn: $(*)$ holds if $n \le 4$.

Thm (Gilboa, H., Slomka): if x is on the main diagonal $x = \lambda(1, \dots, 1)$ then $(*)$ holds.

Cor: Log-M. holds for L = parallelotope, K centered such that $h_K(u_i)/h_K(-u_i)$ is independent of i.

$-1 < t < 1 \qquad \sqrt[n]{\Pr(X < t)} < \sqrt{1 - m_X^2(t)}$

$X = \frac{1}{n} \sum_{i=1}^n U_i$, $U_i \sim \mathrm{Unif}([-1, 1])$ independent

$m_X(t) = \mathbb{E}[X \mid X < t]$

$$\left[(m_X(t) - t) \frac{m_X(t)}{1 - m_X^2(t)} < \frac{1}{n} \quad \left(\overset{?}{\ll} \frac{1}{2n}\right) \right.$$

$\le \frac{c}{n}$