HOME WORK I, PROBABILITY I

1. Prove that an intersection of sigma-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ over Ω is a sigma-algebra.

2. Prove that Borel sigma-algebra on the real line is generated by the collection of sets

$$\{(-\infty, a),\ a \in \mathbb{R}\}.$$

Hint. Show that every open set on the real line can be represented as a disjoint union of open intervals.

3. Show that product of Borel sigma-algebras on $\mathbb{R}$ is the Borel sigma-algebra on $\mathbb{R}^2$.
Hint. Dyadic lattice on $\mathbb{R}^2$ is the collection of squares $[\frac{k}{2^n}, \frac{k+1}{2^n}] \times [\frac{m}{2^n}, \frac{m+1}{2^n}]$, where n runs over non-negative integers, and k and m run over all integers. Prove that every open set in $\mathbb{R}^2$ can be represented as a countable union of dyadic squares.

4. The unit disc T in $\mathbb{R}^2$ is defined as

$$T = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Drop a vertical needle of length 1 onto the unit disc T randomly, so that the center of the needle is distributed uniformly over T. Find the probability that the needle is fully contained in T.

5. Estimate (from above and from below) the probability that the standard Gaussian random variable takes values in $[-1, 5] \cup (10, 13)$.

6. Prove that if X and Y are random variables on Ω with fixed sigma-algebra $\mathcal{F}$, then $X \cdot Y$, and $\max(X, Y)$ are both random variables as well.

7. Let X be standard Gaussian random variable. Find the density of X^2.

8. Consider a measurable space $(\Omega, \mathcal{F}, \mu)$. Prove that for any integrable functions $f,\ g : \Omega \to \mathbb{R}$,
a) $\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu$, for any $a, b \in \mathbb{R}$;
b) $f \leq g$ a.e. $\implies \int f d\mu \leq \int g d\mu$;

c) If $f \geq 0$ a.e. and $\int f d\mu = 0$, then $f = 0$ a.e.

9. Prove that if $f : [0, 1] \to \mathbb{R}$ is Riemann-integrable and Borel measurable, then its Riemann integral coincides with its Lebesgue integral.

10*. Prove that

$$\lim_{n\to\infty} \int g(x) \cos(nx) dx = 0.$$

11***. Let $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$. Prove that for all $a, b, c, d \in \mathbb{R}$ and for every $\lambda \in [0, 1]$, one has

$$\phi^{-1}\left(\frac{1}{\sqrt{2\pi}} \int_{\lambda a+(1-\lambda)c}^{\lambda b+(1-\lambda)d} e^{-\frac{t^2}{2}} dt\right) \geq \lambda\phi^{-1}\left(\frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{-\frac{t^2}{2}} dt\right) + (1-\lambda)\phi^{-1}\left(\frac{1}{\sqrt{2\pi}} \int_{c}^{d} e^{-\frac{t^2}{2}} dt\right).$$

Remark. In particular, any progress in the case $a = -b$, $c = -d$, $c, d > 0$ will be highly appreciated.