Mathematics 1501 Hour Examination  
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Directions: Do all problems. Show your work and justify your answers. This is a closed book examination. Make sure your copy of the examination has four (4) distinct pages; put your name and your recitation leader's name on each page.

1 (20) a. Find all values of $x$ which satisfy $x(x^2 - 25) \geq 0$.

$x(x^2 - 25) = (x+5)(x-5)$  \iff  $x \leq -5$, this product is negative.  \iff  $-5 \leq x \leq 5$, it is positive.  \iff  $0 \leq x \leq 5$, it is negative.  \iff  $5 \leq x$, it is positive.  Thus

$x(x^2 - 25) \geq 0 \iff -5 \leq x \leq 0 \text{ or } 5 \leq x$

b. Find all values of $x$ which satisfy $|3x - 6| < 3$.

$|3x-6| < 3 \iff 3 \ |x-2| < 3 \iff |x-2| < 1 \iff 1 \leq x < 3$

2 (10) Show that the function $f(x) = x \cos(\frac{1}{1+x^2})$ is continuous at zero.

$f(0) = 0$. Also $|x \cos(\frac{1}{1+x^2})| \leq |x|$, so $-|x| \leq x \cos(\frac{1}{1+x^2}) \leq |x|$

Since $|x| \to 0$ as $x \to 0$, the Pinching Theorem shows that

$\lim_{x \to 0} x \cos(\frac{1}{1+x^2}) = 0 = f(0)$. Thus $f$ is continuous at zero.

Since $\cos$ is defined and continuous for all $t$, and $\frac{1}{1+x^2}$ is defined and continuous for all $x$. Thus $\cos(\frac{1}{1+x^2})$ is continuous for all $x$. Since $x$ is continuous for all $x$, the product $x \cos(\frac{1}{1+x^2})$ is continuous at zero.
3. (30) Which of the following limits exist? If the limit does not exist, give a reason why not. If it does exist, evaluate it. You may use the fact (established in class) that 
\[ \lim_{x \to 0} \frac{\sin x}{x} = 1. \]

a. \[ \lim_{x \to 0} \frac{x}{\tan(x)} = \lim_{x \to 0} \frac{x}{\frac{\sin x}{\cos x}} = \lim_{x \to 0} \frac{x}{\sin x} \cos x = \left( \frac{1}{1} \right) 1 = 1 \]

b. \[ \lim_{x \to 3} \frac{2x^2 - 18}{3 - x} = \lim_{x \to 3} \frac{2(x-3)(x+3)}{(3-x)x} = \lim_{x \to 3} \frac{2(x+3)}{x} = \frac{2(6)}{3} = 4 \]

c. \[ \lim_{x \to 0} \frac{\sqrt{x^2 + 36} - 6}{3x^2} = \lim_{x \to 0} \left( \frac{\sqrt{x^2 + 36} - 6}{3x^2} \right) \left( \frac{\sqrt{x^2 + 36} + 6}{\sqrt{x^2 + 36} + 6} \right) \]

\[ = \lim_{x \to 0} \frac{x^2 + 36 - 36}{3x^2 \left( \sqrt{x^2 + 36} + 6 \right)} \]

\[ = \lim_{x \to 0} \frac{x^2}{3x^2 \left( \sqrt{x^2 + 36} + 6 \right)} \]

\[ = \lim_{x \to 0} \frac{1}{3 \left( \sqrt{x^2 + 36} + 6 \right)} = \frac{1}{3 \sqrt{36 + 6}} = \frac{1}{3 (6 + 6)} = \frac{1}{36} \]
4. (30) a. What is the domain of the function \( f(x) = \sqrt{x - 2} + \frac{1}{x^2 - 16} \)?

\[
\text{Domain of } \sqrt{x - 2} = \{ x : x \geq 2 \} \\
\text{Domain of } \frac{1}{x^2 - 16} = \{ x : |x| \neq 4 \} \\
\text{Domain of sum } = \{ x : x \geq 2 \text{ and } x \neq 4 \} = [2, 4) \cup (4, +\infty)
\]

b. Let \( f(x) = \frac{1}{x^2} \) and \( g(x) = \sqrt{x^2 - 4} \). Give explicit formulas for \( f \circ g \) and for \( g \circ f \).

\[
f \circ g (x) = \frac{1}{(\sqrt{x^2 - 4})^2} = \frac{1}{x^2 - 4}
\]

\[
g \circ f (x) = \sqrt{\left(\frac{1}{x^2}\right)^2 - 4} = \sqrt{\frac{1}{x^4} - 4}
\]

c. Find the domain of the function \( f \circ g \) from part b above.

\[
\text{Domain of } g = (-\infty, -2] \cup [2, +\infty) \text{ i.e. } |x| \geq 2
\]

\[
\text{Domain of } f = \{ x : x \neq 0 \}
\]

\[
\text{Domain of } f \circ g = \{ x : |x| \geq 2 \text{ and } \sqrt{x^2 - 4} \neq 0 \}
\]

\[
= \{ x : |x| \geq 2 \text{ and } x^2 \neq 4 \}
\]

\[
= \{ x : |x| > 2 \} = (-\infty, -2) \cup (2, +\infty)
\]
5. (10) Use an epsilon-delta argument (that is, use the definition of limit) to show that 
\[ \lim_{x \to 3} 3x - 5 = -2. \]

Let \( \varepsilon > 0. \)

We want \( |3x - 5 - (-2)| < \varepsilon \), or equivalently, \( |3x - 3| < \varepsilon \).

Now \( |3x - 3| < \varepsilon \) \( \iff \) \( 3|x - 1| < \varepsilon \) \( \iff \) \( |x - 1| < \frac{\varepsilon}{3} \).

Choose \( \delta = \frac{\varepsilon}{3}. \)

Then if \( 0 < |x - 1| < \delta \), then \( |x - 1| < \frac{\varepsilon}{3} \), so \( 3|x - 1| < \varepsilon \), so \( |3x - 3| < \varepsilon \), so \( |3x - 5 - (-2)| < \varepsilon. \)