Mathematics 1501 Hour Examination
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Directions: Do all problems. Show your work and justify your answers. Calculators are not allowed, and this is a closed book examination. You are allowed one prepared sheet. Put your name and your recitation leader’s name on EACH page of your examination.

1 (48) Calculate each of the following derivatives and integrals.

a. \( \int_0^{3/5} (5x-3)^6 \, dx = \int_0^u \frac{6}{5} \, du \)
   \[ u = 5x - 3 \]
   \[ du = 5 \, dx \]
   \[
   = \frac{1}{5} \left. \frac{1}{7} u^7 \right|_0^{3/5} = \frac{-3^7}{35}
   \]
   \[
   = \frac{3^7}{35} = \frac{2187}{35}
   \]

b. \( \int 4x \sqrt{9-x^2} \, dx = \int 4u^{1/2} \, -\frac{du}{2} \)
   \[ u = 9 - x^2 \]
   \[ du = -2x \, dx \]
   \[
   = -2 \int u^{1/2} \, du = \left( -2 \right) \frac{2}{3} u^{3/2} + C = \frac{-du}{2} = x \, dx
   \]
   \[
   = -\frac{4}{3} \left( 9 - x^2 \right)^{3/2} + C
   \]

c. \( \int \tan^3 x \sec^2 3x \, dx = \int u^3 \, \frac{du}{3} = \frac{1}{3} \left( \frac{u}{4} \right)^4 + C \)
   \[ u = \tan 3x \]
   \[ du = 3 \sec^2 3x \, dx \]
   \[ \frac{du}{3} = \sec^2 3x \, dx \]
   \[
   = \frac{1}{12} \tan^4 3x + C
   \]
1. (continued) Calculate each of the following derivatives and integrals.

d. \( \int_{-\pi}^{\pi} \sin x (1 + \cos x) \, dx = 0 \) since the function \( \sin x (1 + \cos x) \) is odd

\[
\begin{align*}
\text{let } u &= 1 + \cos x \\
\frac{du}{dx} &= -\sin x \\
du &= \sin x \, dx \\
\begin{array}{c|c|c|c}
x & u & \frac{du}{dx} & dx \\
\hline
-\pi & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\pi & 2 & 0 & 0
\end{array}
\end{align*}
\]

\[= - \int_0^\pi u \, du = 0 \]

e. \( \frac{d}{dx} \left( \int_0^x \sqrt{1+t} \, dt \right) = \sqrt{1+x} \) since \( \sqrt{1+t} \) is continuous

\[
\begin{align*}
\text{let } u &= 3x \\
\frac{du}{dx} &= 3 \\
du &= 3 \, dx \\
\begin{array}{c|c|c|c}
x & u & \frac{du}{dx} & dx \\
\hline
0 & 0 & 1 & 0 \\
\pi & 3 & 0 & 0 \\
\end{array}
\end{align*}
\]

\[= \left( \sqrt{\tan x} \right)(3) = 3\sqrt{\tan(3x)} \]

2. (14) Let \( A \) be the area of the bounded region enclosed by the parabola \( y = (x-2)^2 \) and the line \( y = 2x + 4 \).

a. Write \( A \) as an integral of the form \( \int [f(x) - g(x)] \, dx \).

\[
\int_0^6 \left[ 2x + 4 \right] - [(x-2)^2] \, dx
\]
2. (continued) Let $A$ be the area of the bounded region enclosed by the parabola $y = (x - 2)^2$ and the line $y = 2x + 4$.

b. Compute the value of the area $A$.

\[
\int_0^6 (2x+4) - (x-2)^2 \, dx = \int_0^6 2x + 4 - x^2 + 4x - 4 \, dx
\]

\[
= \int_0^6 6x - x^2 \, dx = \left[3x^2 - \frac{x^3}{3}\right]_0^6 = 3(36) - 6^3
\]

\[
= 108 - 216 = 108 - 72 = 36
\]

3. (14) A rod is 10 cm. long. It has a density (in g./cm.) at each point $P$ equal to the distance (in cm.) from $P$ to the end of the rod which is closer to $P$.

a. Find the mass of the rod.

By symmetry, the mass is $2 \int_0^5 x \, dx = 2 \left[\frac{x^2}{2}\right]_0^5 = 25$.

b. Find the center of mass of the rod.

By symmetry, $\bar{x}$ of the density function,

the mass is at the center of the rod.

\[
\bar{x} = \frac{1}{25} \left[\int_0^5 x^2 \, dx + \int_0^5 (10-x)x \, dx\right] = ... = 5
\]
4. (24) Let \( R \) denote the bounded region in the plane which lies between the lines \( x = 3 \), \( y = 1 \) and the curve \( y = \sqrt{x} + 1 \).

a. What is the volume of the solid generated by the region \( R \) if \( R \) is revolved about the \( x \)-axis?

\[
V = \int_0^3 \pi \left( (\sqrt{x} + 1)^2 - 1^2 \right) dx = \pi \int_0^3 x + 1 - 1 \, dx = \pi \int_0^3 x \, dx = \pi \left( \frac{3^2}{2} \right)
\]

\[
= \frac{9\pi}{2}
\]

b. What is the volume of the solid generated by the region \( R \) if \( R \) is revolved about the \( y \)-axis?

\[
V = \int_1^3 2\pi x \left( \sqrt{x} + 1 - 1 \right) dx = \int_0^3 2\pi x \sqrt{x} \, dx - \int_0^3 2\pi x \, dx
\]

\[
= \int_0^3 2\pi (u-1) \sqrt{u} \, du - \int_0^3 2\pi x \, dx
\]

\[
= 2\pi \left( \frac{3^3}{3!} - \frac{3}{2} \right) = 2\pi \left( \frac{27}{6} - \frac{3}{2} \right) = 2\pi \left( \frac{9}{2} \right)
\]

\[
= \frac{9\pi}{2}
\]