Spectral Theory, with an Introduction to Operator Means

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Introduction

Spectral Theory

The subject of this work is the spectral theory of linear operators, mostly bounded, on a Hilbert space. As an application of the power of this theory we also give a short introduction to the subject of means (geometric, harmonic, arithmetic, etc.) of positive operators.

Spectral theory is an infinite dimensional version of the diagonalization of a normal matrix. A square matrix (or bounded linear operator) is *normal* if it commutes with its adjoint. It turns out that a matrix A is normal if and only if there exists an orthonormal basis for the underlying (finite dimensional) space that consists entirely of eigenvectors of A. This is equivalent to the existence of a unitary matrix U with the property that U^*AU is diagonal, in which case the diagonal entries of U^*AU are precisely the eigenvalues of A.

If we collect together into a subspace S_λ all of the eigenvectors which correspond to a given eigenvalue λ, then of course A acts on S_λ as multiplication by λ. Putting E_λ equal to the orthogonal projection onto S_λ, we see that A acts on S_λ in the same way as λE_λ. Summing over the finitely many eigenvalues gives us $A = \sum_\lambda \lambda E_\lambda$, so that A has a nice representation in terms of orthogonal projections.

In the case of an infinite dimensional Hilbert space $\mathcal{H}$, even a normal operator may fail to have any eigenvalues, so we need to replace the notion of

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eigenvalue by something more general: a complex number λ lies in the *spectrum* of a bounded operator A if the operator $A - \lambda I$ fails to have an inverse in the algebra of all bounded linear operators on $\mathcal{H}$. It is a consequence of Liouville's Theorem that the spectrum of a bounded linear A is never empty, and we shall show that there is a nice integral representation (an integral with respect to a projection-valued measure on the spectrum of A) which fully generalizes the diagonalization $A = \sum_\lambda \lambda E_\lambda$ of a normal matrix.

This generalized diagonalization enables us to identify any normal bounded linear operator with multiplication by a bounded complex-valued measurable function on a space of square-integrable functions. Since scalar-valued functions commute with one another, normal operators are considerably easier to deal with than general bounded linear operators. This tractability has led both to a rich theory and to a very wide-ranging family of applications. By approximating unbounded operators by bounded ones, we can even extend the applicability of spectral theory to important unbounded operators, such as differentiation.

Our approach to spectral theory is by way of Banach and C^*-algebras, and leads us through the Gelfand theory for commutative Banach algebras and the Gelfand-Naimark characterization of commutative C^*-algebras with identity. This approach puts many of the fundamental theorems for general operator algebras within reach, and for the sake of a broader perspective we include the most basic (von Neumann's Double Commutant Theorem, the Kaplansky Density Theorem, and the Gelfand-Naimark characterization of general C^*-algebras) of these. This inclusion allows us both to give a few further illustrations of the use of spectral theory and to provide an additional

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set of tools that continue to find important applications in many areas of mathematics.

Operator means

In Part II we shall see that a certain representation of a pair of positive operators in terms of commuting functions will lead us to very satisfactory generalizations for operators of the harmonic and geometric means $(2(a^{-1}+b^{-1})^{-1}$ and $(ab)^{1/2}$ respectively) of positive numbers. We shall also see that the subject of means of operators has a rich theory and a set of applications that parallel those of the numerical means, and we shall give interpretations of some related operations in terms of electrical networks. Throughout this second part we take the view that operations on a pair of positive operators can be reduced to similar operations on an associated pair of commuting positive operators, and hence to the same operations on pairs of numerical-valued functions. Finally, we shall establish some connections to the geometry of order intervals in $\mathcal{B}(\mathcal{H})$ and an integral representation for general means of operators.

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Hilbert Space

An *inner product space* is a complex vector space V equipped with a function $\langle\ ,\ \rangle$ on $V \times V$ to $\mathbb{C}$ (the complex numbers) that satisfies the following conditions:

1) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$ for all complex α and β and all x, y and z in V (i.e., $\langle\ ,\ \rangle$ is linear in its first argument);

2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all x and y in V;

3) $\langle x, x\rangle \geq 0$ for all x in V; and

4) $\langle x, x\rangle = 0$ if and only if $x = 0$.

Such a function is called an *inner product* on V. We shall shortly see that each inner product on V leads to a norm on V. A *Hilbert space* is an inner product space that is complete (i.e., every Cauchy sequence is convergent) with respect to the norm induced by its inner product. Note that any function that satisfies 1) and 2) must also be *conjugate-linear* in its second argument, i.e., must satisfy $\langle x, \alpha y + \beta z\rangle = \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle$.

On occasion it is useful to consider a space V equipped with a function on $V \times V$ which satisfies conditions 1), 2), and 3) but not necessarily 4). Such a space is called a *pre-Hilbert space*. It will be convenient for us to have some additional terminology as well: a function $\langle\ ,\ \rangle$ on $V \times V$ to $\mathbb{C}$ which is linear in its first variable and conjugate linear in its second is called a *sesquilinear form* on V; a sesquilinear form is *hermitian* if it satisfies condition 2) above, and *positive* if it satisfies both 2) and 3) above. Thus a pre-Hilbert space is just a complex vector space equipped with a positive sesquilinear form.

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Examples The functions $\langle x, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ and $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx$ are inner products on $\mathbb{C}^n$ and on the space $C[0,1] = \{f : [0,1] \to \mathbb{C} : f$ is continuous$\}$ respectively. For any t in $[0,1]$ the function $\langle f, g\rangle_t = \overline{g(t)}f(t)$ is a positive sesquilinear form on $C[0,1]$.

Our first two propositions follow easily from sesquilinearity by direct calculation (exercise).

Proposition 1 (Polarization Identity) *Let $\langle\ ,\ \rangle$ be a sesquilinear form on V. Then for all x and y in V we have*

$$4\langle x, y\rangle = \sum_{n=0}^{3} i^n \langle x + i^n y, x + i^n y\rangle = \langle x + y, x + y\rangle - \langle x - y, x - y\rangle + \{\langle x + iy, x + iy\rangle - \langle x - iy, x - iy\rangle\}\, i$$

Proposition 2 (Parallelogram Law) *Let $\langle\ ,\ \rangle$ be a sesquilinear form on V. Then for all x and y in V we have*

$$\langle x + y, x + y\rangle + \langle x - y, x - y\rangle = 2\left(\langle x, x\rangle + \langle y, y\rangle\right)$$

Exercise Suppose $\langle\ ,\ \rangle$ is a sesquilinear form on a complex vector space V and that $\langle\ ,\ \rangle$ satisfies condition 3) in the definition of an inner product space (so that $\langle x, x\rangle \geq 0$ for all x in V). Use the polarization identity to show that $\langle\ ,\ \rangle$ is necessarily hermitian. Then show by example that if $\langle\ ,\ \rangle$ is a bilinear form on a *real* vector space V, it can happen that $\langle x, y\rangle$ is not symmetric in x and y even when $\langle x, x\rangle$ is always non-negative. [Hint: consider $Ax \cdot y$ where A is a two by two matrix with real entries and where $\cdot$ denotes the usual inner product on R^2.]

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Observe that if $\langle\ ,\ \rangle$ is a positive sesquilinear form on a complex V, then the function $\|\ \|$ on V to the non-negative reals defined by $\|x\| = \langle x, x\rangle^{1/2}$ satisfies $\|\alpha x\| = |\alpha|\|x\|$ for all α in $\mathbb{C}$ and all x in V. The next result is the key to showing that for any inner product space the function $\|\ \|$ is a norm.

Proposition 3 *If V is a pre-Hilbert space, then for all x and y in V we have the Cauchy-Schwarz inequality:*

$$|\langle x, y\rangle| \le \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}.$$

Moreover, if V is an inner product space, we have equality if and only if the vectors x and y are linearly dependent.

Proof: For all x and y in V and all scalars α and β we have

$$0 \le \|\alpha x - \beta y\|^2 = |\alpha|^2\|x\|^2 - 2\,\mathrm{Re}(\alpha\bar{\beta}\langle x, y\rangle) + |\beta|^2\|y\|^2.$$

Now fix x and y, and choose a real number ϑ so that $e^{i\vartheta}\langle x, y\rangle = |\langle x, y\rangle|$. If both $\|x\|$ and $\|y\|$ are non-zero, put $\alpha = e^{i\vartheta}\|y\|$ and $\beta = \|x\|$ to get

$$0 \le 2\|x\|^2\|y\|^2 - 2\|x\|\,\|y\||\langle x, y\rangle|,$$

and cancel $2\|x\|\,\|y\|$ to get the Cauchy-Schwarz inequality. If either of $\|x\|$ and $\|y\|$ is zero, then consider instead values of α and β such that $\alpha\bar{\beta} = e^{i\vartheta}$. For such values we get

$$2|\langle x, y\rangle| \le |\alpha|^2\|x\|^2 + |\beta|^2\|y\|^2.$$

Letting α go to zero (if $\|y\|$ is zero) or β go to zero (if $\|x\|$ is zero) gives $|\langle x, y\rangle| = 0$, and we have again the Cauchy-Schwarz inequality (with equality holding).

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If x and y are linearly dependent, then it follows easily from the sesquilinearity that $|\langle x, y\rangle| = \|x\|\,\|y\|$. (This holds even in a pre-Hilbert space.) For the converse, suppose that V is an inner product space. We may assume that x and y are non-zero, and for non-zero x and y we have

$$|\langle x, y\rangle| = \|x\|\,\|y\| \quad \text{if and only if } 0 = 2\|x\|^2\|y\|^2 - 2\|x\|\,\|y\||\langle x, y\rangle|.$$

If these equalities hold for some non-zero x and y, then with ϑ as above, and with $\alpha = e^{i\vartheta}\|y\|$ and $\beta = \|x\|$, we have α and β non-zero and

$$\begin{aligned}\|\alpha x - \beta y\|^2 &= |\alpha|^2\|x\|^2 - 2\,\mathrm{Re}(\alpha\bar{\beta}\langle x, y\rangle) + |\beta|^2\|y\|^2 \\ &= 2\|x\|^2\|y\|^2 - 2\|x\|\,\|y\||\langle x, y\rangle| = 0.\end{aligned}$$

In particular, x and y are linearly dependent whenever $|\langle x, y\rangle| = \|x\|\,\|y\|$. □

Corollary 1 (The Triangle Inequality) *In any pre-Hilbert space V we have*

$$\Big|\|x\| - \|y\|\Big| \le \|x + y\| \le \|x\| + \|y\| \quad \textit{for all } x \textit{ and } y,$$

and

$$\|x + y\| = \|x\| + \|y\| \quad \textit{holds if and only if } \langle x, y\rangle = \|x\|\,\|y\|.$$

If V is an inner product space, then $\|x + y\| = \|x\| + \|y\|$ holds if and only if one of the vectors x and y is a non-negative multiple of the other.

Proof: To establish $\|x + y\| \le \|x\| + \|y\|$, expand $\|x + y\|^2 = \langle x + y, x + y\rangle$ to get $\|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2$. Then apply the Cauchy-Schwarz inequality and extract square roots.

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To see that $\|x + y\| = \|x\| + \|y\|$ holds if and only if $\langle x, y\rangle = \|x\|\,\|y\|$ holds, observe that

$$
\begin{aligned}
\|x + y\| = \|x\| + \|y\| \quad &\text{if and only if} \\
\|x + y\|^2 = (\|x\| + \|y\|)^2 \quad &\text{if and only if} \\
\|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2 = \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \quad &\text{if and only if} \\
\mathrm{Re}\langle x, y\rangle = \|x\|\,\|y\|.
\end{aligned}
$$

If now $\langle x, y\rangle = \|x\|\,\|y\|$ holds, then clearly $\mathrm{Re}\langle x, y\rangle = \|x\|\|y\|$, so that $\|x + y\| = \|x\| + \|y\|$. Conversely, if $\|x+y\| = \|x\|+\|y\|$ holds, then $\mathrm{Re}\langle x, y\rangle = \|x\|\,\|y\|$, so that

$$\mathrm{Re}\langle x, y\rangle = \|x\|\,\|y\| \geq |\langle x, y\rangle| \geq \mathrm{Re}\langle x, y\rangle.$$

But this implies that $\|x\|\,\|y\| = |\langle x, y\rangle| = \mathrm{Re}\langle x, y\rangle$, so that $\langle x, y\rangle = |\langle x, y\rangle| = \|x\|\,\|y\|$. Thus we have $\|x\|\,\|y\| = \langle x, y\rangle$ if and only if $\|x + y\| = \|x\| + \|y\|$, as claimed.

To establish $\big|\|x\|-\|y\|\big| \leq \|x+y\|$, apply the inequality $\|u+v\| \leq \|u\|+\|v\|$ with $u = x-y$ and $v = y$ to get $\|x\| - \|y\| \leq \|x-y\|$. Interchanging the roles of x and y gives also $\|y\| - \|x\| \leq \|x - y\|$, so that $\big|\|x\| - \|y\|\big| \leq \|x + y\|$.

Consider now the assertion that $\|x + y\| = \|x\| + \|y\|$ holds if and only if one of the vectors x and y is a non-negative multiple of the other. If $y = \alpha x$ with α non-negative, then

$$
\begin{aligned}
\|x + y\| &= \|(1 + \alpha)x\| = (1 + \alpha)\|x\| \\
&= \|x\| + \alpha\|x\| = \|x\| + \|\alpha x\| = \|x\| + \|y\|.
\end{aligned}
$$

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On the other hand, suppose $\|x+y\| = \|x\| + \|y\|$, and let $\alpha = \|y\|$ and $\beta = \|x\|$. Then since $\langle x, y\rangle = \|x\|\,\|y\|$, we have

$$\begin{aligned}\|\alpha x - \beta y\|^2 &= |\alpha|^2\|x\|^2 - 2\,\mathrm{Re}(\alpha\bar{\beta}\langle x, y\rangle) + |\beta|^2\|y\|^2 \\ &= 2\|x\|^2\|y\|^2 - 2\|x\|\,\|y\|\langle x, y\rangle = 0.\end{aligned}$$

If V is an inner product space, then from $\|\alpha x - \beta y\|^2 = 0$ we get $\alpha x - \beta y = 0$. If now either of x and y is zero, then it is a non-negative multiple of the other; if both are non-zero, then from $\alpha x - \beta y = 0$ we see that each is a strictly positive multiple of the other. □

Corollary 2 *In any pre-Hilbert space V, we have $\big|\|x\| - \|y\|\big| \leq \|x - y\| \leq \|x\| + \|y\|$.*

Proof: Replace y by $-y$ in the previous corollary and use $\|y\| = \|-y\|$. □

It follows from these corollaries that in any pre-Hilbert space the function $\|x\| = \langle x, x\rangle^{1/2}$ is a *seminorm*, i.e., that $\|x\| \geq 0$ for all x, that $\|\alpha x\| = |\alpha|\|x\|$ for all x and all complex α, and that the triangle inequality holds for all x and y. In an inner product space this function is a *norm* (so that also $\|x\| = 0$ if and only if $x = 0$). It follows that all topological notions (convergence, open and closed, etc.) are available in any pre-Hilbert space.

Corollary 3 *In any pre-Hilbert space, if $\{x_\gamma\}$ converges to x and $\{y_\gamma\}$ converges to y, then $\langle x_\gamma, y_\gamma\rangle$ converges to $\langle x, y\rangle$ and $\|x_\gamma\|$ converges to $\|x\|$.*

Proof: We have $|\langle x, y\rangle - \langle x_\gamma, y_\gamma\rangle| \leq |\langle x, y - y_\gamma\rangle| + |\langle x - x_\gamma, y_\gamma\rangle| \leq \|x\|\|y - y_\gamma\| + \|x - x_\gamma\|\|y_\gamma\|$. The first result now follows from the fact that the tail end of a convergent net (or sequence) is bounded. The result on seminorms follows easily from the inequality $\big|\|x\| - \|y\|\big| \leq \|x - y\|$. □

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Remark 1 For the reader unacquainted with nets there is no real loss in reading this corollary just as a statement about sequences, which are good enough for most purposes in any normed space. The formulation of the last proof may seem a bit odd since a convergent sequence is always bounded, but convergent nets do not in general have this property. What is true (and sufficient for this proof) is that whenever the net $\{y_\gamma\}$ is convergent, each subnet of the form $\{y_\gamma : \gamma \geq \gamma_0\}$ is bounded: this too follows from the usual corollary $\|x - y\| \geq |\|x\| - \|y\||$ to the triangle inequality.

Definition 1 Suppose $\langle \, , \rangle$ is a positive sesquilinear form on V. If $\langle x, y \rangle = 0$, we say that x and y are *orthogonal*. If S is any subset of V, then the set $S^\perp = \{x \in V : \forall \, y \in S, \langle x, y \rangle = 0\}$ of all elements of V that are orthogonal to every element of S is called the *orthogonal complement* of S.

Recall that the closed linear span of a set S is the closure of the span of S, i.e., the closure of the set of all linear combinations of elements of S. It is easy to see (see the exercises) that the closure of a subspace of V is again a subspace of V, so the closed linear span of S is always a subspace of V. It is clearly the *smallest* closed subspace of V that contains S: if $S \subseteq K$ and K is a closed subspace of V, then K contains the closed linear span of S.

Exercise Let S and T be subsets of V. Then $S^\perp$ is always a closed subspace, $S \subseteq S^{\perp\perp}$, and $S \subseteq T$ implies $S^\perp \supseteq T^\perp$. Moreover, the closed linear span M of S satisfies $M^\perp = S^\perp$ and $M \subseteq S^{\perp\perp}$.

Note that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ if and only if $\text{Re}\langle x, y \rangle = 0$, so that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ whenever x and y are orthogonal.

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Proposition 4 *Suppose $\langle\,,\,\rangle$ is a positive sesquilinear form on V. Then for any x in V, we have $\langle x,x\rangle = 0$ if and only if $\langle x,y\rangle = 0$ for all y in V. The set $\{x \in V : \langle x,x\rangle = 0\}$ is a closed subspace of V.*

Proof: The equivalence $\langle x,x\rangle = 0$ if and only if $\langle x,y\rangle = 0$ for all y follows directly from the Cauchy-Schwarz Inequality. For any y in V the set $\{x \in V : \langle x,y\rangle = 0\}$ is easily seen to be a closed subspace, so the intersection of all such sets over all y is also a closed subspace. □

Although we have defined an inner product only on a complex vector space, real inner product spaces can also be defined, and the geometry of such spaces is Euclidean. Indeed, the assertion that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ whenever x and y are orthogonal is just the Pythagorean Theorem. Another important Euclidean property enjoyed by Hilbert spaces is the following result. Recall that a subset C of a real vector space is *convex* if whenever C contains x and y, then also C contains the line segment connecting them.

Theorem 1 (Nearest Point Theorem) *Let C be a non-void closed convex subset of a Hilbert space $\mathcal{H}$, and let x_0 be a point in $\mathcal{H}$. Then there exist a point z of C such that $\|x_0 - z\| = \inf\{\|x_0 - c\| : c \in C\}$. Such a point z is unique and satisfies $\mathrm{Re}\langle z, x_0 - z\rangle \geq \mathrm{Re}\langle x, x_0 - z\rangle$ for all x in C.*

Proof: Let $d = \inf\{\|x_0 - c\| : c \in C\}$. Choose a sequence $\{x_n\}$ in C with $\|x_0 - x_n\|$ converging to d. Then since $\frac{x_n + x_m}{2}$ lies in C, the parallelogram identity gives us

$$\begin{aligned}\|x_n - x_m\|^2 &= \|(x_n - x_0) - (x_m - x_0)\|^2 \\ &= 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - \|(x_n - x_0) + (x_m - x_0)\|^2\end{aligned}$$

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$$
\begin{aligned}
&= 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4\left\|\frac{x_n + x_m}{2} - x_0\right\|^2 \\
&\leq 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4d^2
\end{aligned}
$$

Since $\|x_n - x_0\| \to d$, the sequence $\{x_n\}$ must be Cauchy. Since $\mathcal{H}$ is complete, $\{x_n\}$ converges to some z in $\mathcal{H}$. But C is closed, so z lies in C. For any such z,

$$\|x_n - x_0\| \to \|z - x_0\|,$$

so $\|z - x_0\| = d$. If $\|z_1 - x_0\| = d = \|z_2 - x_0\|$ with $z_i \in C$, then

$$
\begin{aligned}
\|z_1 - z_2\|^2 &= \|(z_1 - x_0) - (z_2 - x_0)\|^2 \\
&= 2\|z_1 - x_0\|^2 + 2\|z_2 - x_0\|^2 - 4\left\|\frac{z_1 + z_2}{2} - x_0\right\|^2 \\
&\leq 2d^2 + 2d^2 - 4d^2 = 0,
\end{aligned}
$$

so $z_1 = z_2$. □

If now z is as above, x is in C, and t lies in $[0, 1]$, then $z + t(x - z) = tx + (1 - t)z$ lies in C, so that

$$
\begin{aligned}
\|x_0 - z\|^2 \leq \|x_0 - (z + t(x - z))\|^2 &= \|x_0 - z - t(x - z)\|^2 \\
&= \|x_0 - z\|^2 + t^2\|x - z\|^2 - 2t\,\mathrm{Re}\langle x_0 - z, x - z\rangle,
\end{aligned}
$$

so that

$$0 \leq t^2\|x - z\|^2 - 2t\,\mathrm{Re}\langle x_0 - z, x - z\rangle.$$

For t in $(0, 1]$ this gives

$$0 \leq t\|x - z\|^2 - 2\,\mathrm{Re}\langle x_0 - z, x - z\rangle.$$

Letting t tend to zero gives

$$0 \leq -2\,\mathrm{Re}\langle x_0 - z, x - z\rangle,$$

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from which the inequality $\text{Re}\langle z, x_0 - z\rangle \geq \text{Re}\langle x, x_0 - z\rangle$ follows easily.

Corollary 4 *Let M be a closed subspace of a Hilbert space $\mathcal{H}$. Let x_0 be a point of $\mathcal{H}$. Then there exists a unique point m of M that is closest to x_0, and $x_0 - m$ is orthogonal to every element of M.*

Proof: The existence and uniqueness of m follow immediately from the theorem. Suppose $\langle m', x_0 - m\rangle \neq 0$ for some m' in M. Then as λ ranges through $\mathbb{C}$, so does $\langle \lambda m', x_0 - m\rangle$. Thus $\{\text{Re}\langle x, x_0 - m\rangle : x \in M\}$ is not bounded. But for any x in M we have $\text{Re}\langle m, x_0 - m\rangle \geq \text{Re}\langle x, x_0 - m\rangle$. Thus $\langle m', x_0 - m\rangle = 0$ for every m' in M. □

Corollary 5 *If M is a closed subspace of a Hilbert space $\mathcal{H}$, then each element of $\mathcal{H}$ can be expressed uniquely as a sum $x = m + m^\perp$, where m lies in M and $m^\perp$ is orthogonal to each element of M. Moreover, $\|x\|^2 = \|m\|^2 + \|m^\perp\|^2$, and m is the unique point in M which is closest to x.*

Proof: Let $m^\perp = x - m$, where m is the point of M closest to x. Then $x = m + m^\perp$, and $m^\perp$ is orthogonal to M. If also $x = m_1 + m_2$, where m_1 and m_2 lie in M and $M^\perp$ respectively, then $m_1 - m = m^\perp - m_2$ lies in M and in $M^\perp$. Since then $m_1 - m$ is orthogonal to itself, it must be zero, so m and $m^\perp$ are unique. The equation involving the squares of the norms is just the Pythagorean Theorem applied to x, m and $m^\perp$. □

Definition 2 The point m is called the *orthogonal projection* of x onto M. By the last corollary, it is uniquely determined by the conditions "m lies in M" and "$x - m$ is orthogonal to every point of M."

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Exercise Show that the Nearest Point Theorem and its corollaries imply that for any subset S of any Hilbert space $\mathcal{H}$, $S^{\perp\perp}$ is always the closed linear span of S. Conclude that $S = S^{\perp\perp}$ if and only if S is a closed subspace of $\mathcal{H}$, that $S^{\perp} = S^{\perp\perp\perp}$ for any subset S of $\mathcal{H}$, and that for closed subspaces S and T of $\mathcal{H}$ we have $S \subseteq T$ if and only if $S^{\perp} \supseteq T^{\perp}$.

Thus closed subspaces of a Hilbert space always occur in pairs that are mutually orthogonal and sum to the whole space. Since $M = M^{\perp\perp}$ for any closed subspace M, the relationship between a closed subspace and its orthogonal complement is completely symmetrical. Note that the orthogonal complement of $\mathcal{H}$ is $\{0\}$.

A set S of elements of $\mathcal{H}$ is an *orthogonal family* if any two distinct elements of S are orthogonal. An orthogonal family is an *orthonormal family* if each of its elements has norm one.

Note that if S is an orthogonal family and S does not contain the zero vector, then S is linearly independent: if $0 = \sum_{i=1}^{n} \alpha_i s_i$ for some finite set $\{s_1, s_2, \ldots, s_n\}$ of distinct elements of S, then for each i_0 between 1 and n, $0 = \langle \sum_{i=1}^{n} \alpha_i s_i, s_{i_0} \rangle = \alpha_{i_0} \langle s_{i_0}, s_{i_0} \rangle$, and $\langle s_{i_0}, s_{i_0} \rangle \neq 0$. Moreover, for such a linear combination we have

$$\left\| \sum_{i=1}^{n} \alpha_i s_i \right\|^2 = \left\langle \sum_{i=1}^{n} \alpha_i s_i, \sum_{i=1}^{n} \alpha_i s_i \right\rangle = \sum_{i=1}^{n} |\alpha_i|^2 \|s_i\|^2;$$

in particular, if S is an orthonormal family, we get

$$\left\| \sum_{i=1}^{n} \alpha_i s_i \right\|^2 = \sum_{i=1}^{n} |\alpha_i|^2.$$

Suppose now that S is an orthonormal family and that $\{s_1, s_2, \ldots, s_n\}$ is a set of distinct elements of S. If s lies in the span M_0 of $\{s_1, s_2, \ldots, s_n\}$,

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then $s = \sum_{i=1}^n \alpha_i s_i$ implies that $\langle s, s_i\rangle = \alpha_i$ for every i, so $s = \sum_{i=1}^n \langle s, s_i\rangle s_i$ and $\|s\|^2 = \|\sum_{i=1}^n \alpha_i s_i\|^2 = \sum_{i=1}^n |\langle s, s_i\rangle|^2$. More generally, if x is in $\mathcal{H}$, then $x_0 = \sum_{i=1}^n \langle x, s_i\rangle s_i$ lies in M_0, and by direct calculation $x - x_0$ is orthogonal to each s_i. It follows that x_0 is the orthogonal projection of x onto M_0. Moreover,

$$\|x\|^2 = \|x_0\|^2 + \|x - x_0\|^2 = \sum_{i=1}^n |\langle x, s_i\rangle|^2 + \|x - x_0\|^2 \geq \sum_{i=1}^n |\langle x, s_i\rangle|^2.$$

Thus for any finite subset F of S, we have

$$\|x\|^2 \geq \sum_{s\in F} |\langle x, s\rangle|^2 = \left\|\sum_{s\in F} \langle x, s\rangle s\right\|^2.$$

We take the meaning of the expression $\sum_{s\in S} |\langle x, s\rangle|^2$ to be $\sup\big\{\sum_{s\in F} |\langle x, s\rangle|^2 : F \subseteq S \text{ and } F \text{ is finite}\big\}$, so we have the following result.

Theorem 2 (Bessel's Inequality) *Whenever S is an orthonormal family in a Hilbert space $\mathcal{H}$, we have for each x in $\mathcal{H}$ that*

$$\|x\|^2 \geq \sum_{s\in S} |\langle x, s\rangle|^2.$$

Definition 3 The expressions $\langle x, s\rangle$ where s ranges through S, are called the *Fourier coefficients of* x with respect to the orthonormal family S.

Remark 2 Whenever an indexed family $\{t_s\}_{s\in S}$ of non-negative real numbers has a finite sum, there can be only countably many s such that $t_s > 0$. (Each of the subsets $S_n = \{s \in S : t_s > 1/n\}$ is finite, and $\{s \in S : t_s > 0\}$ is the union of the countably many sets S_n.) It thus follows from Bessel's Inequality that for any given x and any given orthonormal set S in $\mathcal{H}$, only countably many of the Fourier coefficients of x with respect to S can be non-zero.

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Now let S be any orthonormal family in $\mathcal{H}$, and let M be the closed linear span of S. Then for any element x of $\mathcal{H}$ and any subset F (finite or infinite) of S we have

$$\sum_{s\in F} |\langle x, s\rangle|^2 \leq \sum_{s\in S} |\langle x, s\rangle|^2 \leq \|x\|^2.$$

If now $\epsilon > 0$ is given, we may choose a finite subset F of S such that $\sum_{s\in S\setminus F} |\langle x, s\rangle|^2 < \epsilon^2$. Then for all finite subsets F_1 and F_2 that contain F, we have $\| \sum_{s\in F_1} \langle x, s\rangle s - \sum_{s\in F_2} \langle x, s\rangle s\| = \left(\sum_{s\in\Delta} |\langle x, s\rangle|^2\right)^{1/2} < \epsilon$, where $\Delta = (F_1\setminus F_2)\cup(F_2\setminus F_1)$ is the symmetric difference of F_1 and F_2. It follows that the partial sums of $\sum_{s\in S}\langle x, s\rangle s$ are norm-Cauchy, and hence that $\sum_{s\in S}\langle x, s\rangle s$ sums in $\mathcal{H}$ to an element z of M. By the continuity of the inner product, $\langle z, s\rangle = \langle x, s\rangle$ for each s in S, so $x - z$ is orthogonal to M, so z is the orthogonal projection of x onto M.

Now by the sesquilinearity and continuity of the inner product and by the fact that S is an orthonormal set, we have

$$\begin{aligned}
\|x - z\|^2 &= \left\langle x - \sum_{s\in S}\langle x, s\rangle s, x - \sum_{s\in S}\langle x, s\rangle s \right\rangle \\
&= \langle x, x\rangle - 2\,\mathrm{Re}\left\langle x, \sum_{s\in S}\langle x, s\rangle s \right\rangle \\
&\qquad + \left\langle \sum_{s\in S}\langle x, s\rangle s, \sum_{s\in S}\langle x, s\rangle s \right\rangle \\
&= \langle x, x\rangle - 2\sum_{s\in S} |\langle x, s\rangle|^2 \\
&\qquad + \sum_{s\in S} |\langle x, s\rangle|^2 = \|x\|^2 - \sum_{s\in S} |\langle x, s\rangle|^2.
\end{aligned}$$

It follows that x lies in M if and only if $\|x\|^2 = \sum_{s\in S} |\langle x, s\rangle|^2$. In particular, $\|x\|^2 = \sum_{s\in S} |\langle x, s\rangle|^2$ for every $x \in \mathcal{H}$ if and only if the closed linear span M

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of S is all of $\mathcal{H}$.

Remark 3 When x lies in M, the equation $x = \sum_{s\in S}\langle x, s\rangle s$ is called the *Fourier expansion of x with respect to the orthonormal system S*. In our treatment here the system S is unordered, but the collection $\mathcal{F}$ of all finite subsets of S is partially ordered by set theoretic inclusion. In fact, this relation actually directs the set $\mathcal{F}$ (see [CGL]): for each pair F_1, F_2 in $\mathcal{F}$, there exists F_3 ($= F_1 \cup F_2$) in $\mathcal{F}$ with $F_1 \subseteq F_3$ and $F_2 \subseteq F_3$. The reader acquainted with nets will recognize the functions on $\mathcal{F}$ given by

$$F \to \sum_{s\in F}\langle x, s\rangle s$$

and

$$F \to \sum_{s\in F} |\langle x, s\rangle|^2$$

as nets that converge respectively to $\sum_{s\in S}\langle x, s\rangle s$ and $\sum_{s\in S} |\langle x, s\rangle|^2$. Any reader not familiar with nets may take the following as the definition of the unordered sum $\sum_{s\in S} t_s$ of a collection $\{t_s\}_{s\in S}$: we say that $\sum_{s\in S} t_s$ *sums to* t in a normed space V if for every $\epsilon > 0$ there exists a finite subset F_ϵ of S such that whenever $F_\epsilon \subseteq F \subseteq S$ and F is finite, then also $\|t - \sum_{s\in F} t_s\| < \epsilon$. We say that $\{\sum_{s\in F} t_s : F \subseteq S \text{ and } F \text{ is finite}\}$ is *Cauchy* if for every $\epsilon > 0$ there exists a finite subset F_ϵ of S with the following property: whenever F_1 and F_2 are finite subsets of S with $F_\epsilon \subseteq F_1$ and $F_\epsilon \subseteq F_2$, then also $\| \sum_{s\in F_1} t_s - \sum_{s\in F_2} t_s\| < \epsilon$. It can be shown (see for example [KJL, pp. 190–194]) that V is complete if and only if every Cauchy net in V is convergent, and that if $t_s \geq 0$ (in $\mathbb{R}$) for all $s \in S$, then $\sum_{s\in S} t_s$ is the supremum of its partial sums $\sum_{s\in F} t_s$, where F ranges over all finite subsets of S. It is

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relatively easy to see that if S is the set of all positive integers, then the unordered sum $\sum_{s\in S} t_s$ agrees with the usual ordered sum $\sum_{s=1}^{\infty} t_s$. For an approach to Fourier expansions that avoids nets, see Exercise 8 below.

Definition 4 An orthonormal family S is said to be *complete* if it has the property that for each x in $\mathcal{H}$, $\|x\|^2 = \sum_{s\in S} |\langle x, s\rangle|^2$. In this case we also say that S is a *Hilbert space basis* for $\mathcal{H}$. The equation $\|x\|^2 = \sum_{s\in S} |\langle x, s\rangle|^2$ is called the *Parseval Identity*.

Thus an orthonormal family S is complete if and only if its closed linear span is $\mathcal{H}$ (i.e., if and only if $S^{\perp} = \{0\}$) if and only if for each x in $\mathcal{H}$ we have $x = \sum_{s\in S}\langle x, s\rangle s$.

Exercise An orthonormal family is said to be *maximal* if it is not properly contained in any other orthonormal family. Show that an orthonormal family is maximal if and only if it is complete. Then use Zorn's Lemma to show that every Hilbert space has a complete orthonormal family. (Order the orthonormal families by set-theoretic inclusion.)

Thus every Hilbert space has a basis, and the basis for a Hilbert space plays a role analogous to the role played by a (vector space) basis in a vector space. This suggests that there might be a notion of Hilbert space dimension.

Definition 5 The *Hilbert space dimension* of $\mathcal{H}$ is the cardinality of any basis for $\mathcal{H}$.

For this notion to be unambiguously defined we need the following result.

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Proposition 5 *Any two bases for the same Hilbert space have the same cardinality.*

Proof: Let S and T be bases for $\mathcal{H}$, and let α and β be their cardinal numbers respectively. If α is finite, then S is a basis for the vector space $\mathcal{H}$, so no set of vectors in $\mathcal{H}$ of cardinality greater than α can be linearly independent. Since T is linearly independent, we have $\beta \leq \alpha$. But then S and T are both bases for a finite dimensional vector space, and hence $\beta = \alpha$. Similarly, if β is finite, then $\beta = \alpha$.

Assume then that α and β are infinite cardinal numbers. For each s in S, let $T_s = \{t \in T : \langle t, s \rangle \neq 0\}$. By Bessel's Inequality, each T_s is countable. Since each t in T is nonzero and S is a basis, T is contained in the union of all the T_s. It follows that the cardinal number β of T is at most $\alpha\aleph_0 = \alpha$. Similarly, α is at most β, so $\alpha = \beta$, and the proposition is proved. □

Theorem 3 *Two Hilbert spaces are isometrically isomorphic if and only if they have the same dimension.*

Lemma 1 *A linear map U between two Hilbert spaces is isometric if and only if it satisfies $\langle Ux, Uy \rangle = \langle x, y \rangle$ for every x and y in its domain. (For real Hilbert spaces, the analogous result shows that a linear bijection between real Hilbert spaces preserves angles whenever it preserves norms.)*

Proof: If $\langle Ux, Uy \rangle = \langle x, y \rangle$ holds for all x and y in the domain of U, then in particular with $x = y$ we see that Ux and x always have the same norm. If conversely U is isometric, then it follows from the polarization identity that $\langle Ux, Uy \rangle = \langle x, y \rangle$ holds for all x and y in the domain of U. □

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Proof of Theorem 3 Suppose U is an isometric isomorphism between two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and let S be a basis for $\mathcal{H}$. Now U preserves inner products, so it carries each orthonormal family in $\mathcal{H}$ to an orthonormal family in $\mathcal{K}$. Thus since U is continuous and linear, $x = \sum_{s\in S}\langle x, s\rangle s$ implies $Ux = \sum_{s\in S}\langle x, s\rangle Us = \sum_{s\in S}\langle Ux, Us\rangle Us$. It follows that $\{Us : s \in S\}$ is a basis for the image of U, i.e., for $\mathcal{K}$. The restriction of U to S is clearly one-to-one, so the sets S and $U(S)$ have the same cardinality, and $\mathcal{H}$ and $\mathcal{K}$ must therefore have the same dimension.

Conversely, let S and T be bases for $\mathcal{H}$ and $\mathcal{K}$ respectively, and suppose that J is a bijection from S to T. Then we may extend J to a linear bijection from the span of S to the span of T by setting $J\left(\sum \alpha_i s_i\right) = \sum \alpha_i J(s_i)$, and we have

$$\left\|J\left(\sum \alpha_i s_i\right)\right\|^2 = \left\|\sum \alpha_i J(s_i)\right\|^2 = \sum |\alpha_i|^2 = \left\|\sum \alpha_i s_i\right\|^2,$$

since $J(S)$ and S are orthonormal families. Thus J is isometric on the span of S. It is easy to check that J has a unique extension to a linear isometry from the closure of the span of S to the closure of the span of T, i.e., from $\mathcal{H}$ to $\mathcal{K}$.

It follows from the theorem and lemma that two Hilbert spaces have the same structure as Hilbert spaces if and only if they have the same Hilbert space dimension. Many of the theorems from classical analysis can be interpreted as statements that certain pairs are isomorphic as Hilbert spaces, or that certain maps (e.g., the Fourier transform) carry a basis for one Hilbert space to a basis for another.

Exercise Show that a Hilbert space is separable (as a metric space) if and

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only if it has a countable (Hilbert space) basis. (Consider a family of pairwise disjoint open balls, each centered at one of the basis elements.)

Thus, up to Hilbert space isomorphism, there is exactly one separable infinite dimensional Hilbert space. The spaces $L_2[0,1]$, $L_2(\mathbb{R})$, and ℓ^2 from the theory of integration are all complete (by the Riesz-Fischer Theorem) inner product spaces, and so are manifestations of this unique separable infinite dimensional space.

Exercises

1. Prove the Polarization Identity.

2. Prove the Parallelogram Law.

3. (Topology with seminorms) Let $\| \ \|$ be a seminorm on a complex vector space V. We say that a subset S of V is *open* if for each $x \in S$ there exists $\epsilon > 0$ such that $B_\epsilon(x) = \{z \in V : \|z - x\| < \epsilon\} \subseteq S$. A subset of V is *closed* if its complement in V is open. If $\{x_n\}$ is a sequence in V, we say that $\{x_n\}$ *converges to* x if for every $\epsilon > 0$ there exists N such that $n \geq N$ implies that $\|x_n - x\| < \epsilon$. The *closure* $\bar{S}$ of S is the intersection of all the closed subsets of V that contain S.

   a) Show that any union of open subsets of V is again open, and that any intersection of closed subsets of V is again closed.

   b) Show that $x \in \bar{S}$ if and only if there exists a sequence $\{x_n\}$ in S such that $\{x_n\}$ converges to x.

   c) Show that if S is a subspace of V, then $\bar{S}$ is also a subspace of V.

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d) Conclude that if $\langle\ \rangle$ is a positive sesquilinear form on a complex V and if $S \subseteq V$, then the closed linear span of S is the smallest closed vector subspace of V that contains S.

4. Suppose $\langle\ ,\ \rangle$ is a sesquilinear form on a complex vector space V and that $\langle\ ,\ \rangle$ satisfies condition 3) in the definition of an inner product space (so that $\langle x, x\rangle \geq 0$ for all x in V). Use the polarization identity to show that $\langle\ ,\ \rangle$ is necessarily hermitian. Then show by example that if $\langle\ ,\ \rangle$ is a bilinear form on a *real* vector space V, it can happen that $\langle x, y\rangle$ is not symmetric in x and y even when $\langle x, x\rangle$ is always non-negative. [Hint: consider $Ax \cdot y$ where A is a two by two matrix with real entries and where $\cdot$ denotes the usual inner product on R^2.]

5. Suppose V is equipped with a positive sesquilinear form, and let S and T be subsets of V. Show that $S^\perp$ is always a closed subspace, that $S \subseteq S^{\perp\perp}$, and that $S \subseteq T$ implies $S^\perp \supseteq T^\perp$. Show also that the closed linear span M of S satisfies $M^\perp = S^\perp$ and $M \subseteq S^{\perp\perp}$.

6. Suppose $\mathcal{H}$ is a Hilbert space. Use the Nearest Point Theorem to show that for any subset S of $\mathcal{H}$, $S^{\perp\perp}$ is always the closed linear span of S. Conclude that $S = S^{\perp\perp}$ if and only if S is a closed subspace of $\mathcal{H}$, that $S^\perp = S^{\perp\perp\perp}$ for any subset S of $\mathcal{H}$, and that for closed subspaces S and T of $\mathcal{H}$ we have $S \subseteq T$ if and only if $S^\perp \supseteq T^\perp$.

7. An orthonormal family is said to be *maximal* if it is not properly contained in any other orthonormal family. Show that an orthonormal family is maximal if and only if it is complete. Then use Zorn's Lemma

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to show that every Hilbert space has a complete orthonormal family. (Order the orthonormal families by set-theoretic inclusion.)

8. (Fourier expansions without the use of nets) Let S be an orthonormal family in $\mathcal{H}$, and let $x \in \mathcal{H}$. Let $\{s_1, s_2, \dots\}$ be an enumeration of the countable family of all those elements s of S for which $\langle x, s\rangle \neq 0$. Suppose $\{s_1, s_2, \dots\}$ is infinite.

   a) Show that $\sum_{i=1}^{+\infty} |\langle x, s_i\rangle|^2 = \sup\Big\{\sum_{s\in F} |\langle x, s\rangle|^2 : F \subseteq S$ and F is finite$\Big\}$ $(= \sum_{s\in S} |\langle x, s\rangle|^2)$.

   b) Show that the partial sums of the series $\sum_{i=1}^{+\infty} \langle x, s_i\rangle s_i$ converge to an element z of $\mathcal{H}$, and that z is the orthogonal projection of x onto the closed linear span of S.

   c) Show that x lies in the closed linear span of S if and only if $\|x\|^2 = \sum_{i=1}^{+\infty} |\langle x, s_i\rangle|^2$, i.e., if and only if $\|x\|^2 = \sum_{s\in S} |\langle x, s\rangle|^2$.

9. Show that a Hilbert space is separable (as a metric space) if and only if it has a countable basis. (Consider a family of pairwise disjoint open balls, each centered at one of the basis elements.)

10. A Banach space is said to be *uniformly convex* if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$ imply $\|\frac{1}{2}(x + y)\| < 1 - \delta(\epsilon)$. Prove that the closest point theorem is true in a uniformly convex space.

11. Give an example to show that even if the "existence" portion of the closest point theorem holds in a Banach space, the "uniqueness" part need not hold.

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12. Give an example of a non-empty closed convex set C in a Banach space such that there is no point z with $\|z\| = \inf\{\|x\| : x \in C\}$.

13. Let V be a Banach space. Show that the parallelogram law holds in V if and only if there exists an inner product on V that induces the norm of V. (That is, a Banach space in which the parallelogram law holds is a Hilbert space.)

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Linear Maps and Linear Functionals

Let V and W be vector spaces over a field K. Recall that a function (or "map," or "mapping") T from V to W is *linear* if $T(ax+by) = aT(x)+bT(y)$ for all x and y in V and all a and b in K. It is easy to check that if aT and $S+T$ are defined by $(aT)(x) = a(T(x))$ and $(S+T)(x) = S(x)+T(x)$, then the set of all linear functions on V to W is again a vector space over K. A linear map from V into K is called a *linear functional* on V, and the space of all linear functionals on V is called the *algebraic dual* space of V.

Suppose now that V and W are normed vector spaces (over $\mathbb{C}$ or $\mathbb{R}$). A linear function T from V to W is *bounded* if there exists a non-negative real number M such that for all x in V we have $\|T(x)\| \leq M\|x\|$. If T is bounded, then the infimum $\|T\|$ of all such M also satisfies $\|T(x)\| \leq \|T\| \, \|x\|$ and is called the *norm* of T. When $\|T\| \leq 1$, we say that T is *norm-decreasing*.

It is easy to check that the function $\| \ \|$ actually is a norm on the space $\mathcal{B}(V,W)$ of all bounded linear maps from V into W. An elementary result from functional analysis (exercise) asserts that if W is complete (i.e., if W is a *Banach space*), then $\mathcal{B}(V,W)$ is also complete. In particular, $\mathcal{B}(V,\mathbb{C})$ is always a Banach space. The Banach space $\mathcal{B}(V,\mathbb{C})$ is called the *Banach space dual*, or *continuous dual*, of the normed space V. It is generally denoted by V' or V^*. Because of other uses of these same symbolisms, neither of these notations is ideal. We will use whichever is convenient for the context.

Proposition 6 *Let T be a bounded linear map from a normed space V to a*

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normed space W. Suppose V is not the zero space. Then

$$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} : x \neq 0\right\} = \sup\{\|T(x)\| : \|x\| = 1\}$$
$$= \sup\{\|T(x)\| : \|x\| \leq 1\}.$$

Proof: Exercise. □

Remark 4 If T is linear from V into W, and if any one of these suprema is finite, then T is bounded.

The norm on $\mathcal{B}(V, W)$ has another useful property: whenever a composition ST of two bounded linear maps is defined, it satisfies $\|ST(x)\| = \|S\|\,\|T(x)\| \leq \|S\|\,\|T\|\,\|x\|$ for all x, so that $\|ST\| \leq \|S\|\,\|T\|$. In particular, for S and T in $\mathcal{B}(V, V)$ the composition is always defined and we always have $\|ST\| \leq \|S\|\,\|T\|$. We shall generally denote $\mathcal{B}(V, V)$ by $\mathcal{B}(V)$. Note that $\mathcal{B}(V)$ is a ring with identity I, where I is defined by $I(x) = x$ for all x in V, and that the norm of the identity operator is one.

Suppose now that $\mathcal{H}$ is a Hilbert space, and that y is a vector in $\mathcal{H}$. The Cauchy-Schwarz inequality implies that the linear functional $\varphi_y(x) = \langle x, y\rangle$ on $\mathcal{H}$ is bounded. In fact, since $\varphi_y(y) = \|y\|^2$, it is easy to see that the norm of φ_y is equal to the norm of y. The Riesz Representation theorem tells us that every bounded linear functional on a Hilbert space arises in this way.

Theorem 4 (Riesz Representation Theorem for Hilbert Space) *Let φ be a bounded linear functional on a Hilbert space $\mathcal{H}$. Then there exists a unique vector y in $\mathcal{H}$ such that $\varphi(x) = \langle x, y\rangle$ for all x in $\mathcal{H}$. Moreover, the norm of y agrees with the norm of φ.*

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Proof: We may assume that $\mathcal{H}$ is not the zero space and that φ is non-zero. Since φ is continuous and non-zero, its kernel K is a proper closed linear subspace of $\mathcal{H}$. Choose a vector z of length one in the orthogonal complement of K, and put $y = \overline{\varphi(z)}z$. Note that, as z is not in K, $\varphi(y) \neq 0$. Since

$$\varphi(y) = \varphi(\overline{\varphi(z)}z) = \overline{\varphi(z)}\varphi(z) = \langle\, \overline{\varphi(z)}z, \overline{\varphi(z)}z\rangle = \langle y, y\rangle,$$

it follows that the functional φ agrees with the functional $\varphi_y(x) = \langle x, y\rangle$ on K and at y, and hence on the span of $K \cup \{y\}$. Now for any x in $\mathcal{H}$, we have $\varphi\left(x - \frac{\varphi(x)}{\varphi(y)}\, y\right) = 0$, so the span of $K \cup \{y\}$ is all of $\mathcal{H}$. Thus the desired vector exists. Since $\varphi = \varphi_y$, $\|\varphi\| = \|\varphi_y\| = \sup_{\|x\|\leq 1} |\langle x, y\rangle| \leq \|y\|$. But $\sup_{\|x\|\leq 1} |\langle x, y\rangle| \geq \left\langle \frac{y}{\|y\|}\,, y \right\rangle = \|y\|$, so $\|\varphi\| = \|y\|$. To see that y is unique, observe that if $\langle x, y\rangle = \langle x, w\rangle$ for all x in $\mathcal{H}$, then $y - w$ is orthogonal to all of $\mathcal{H}$. □

Corollary 6 *For any vector x in a Hilbert space $\mathcal{H}$, and any bounded linear operator T in $\mathcal{B}(\mathcal{H})$, we have $\|x\| = \sup_{\|y\|\leq 1} |\langle x, y\rangle|$ and $\|T\| = \sup_{\|x\|\leq 1\,, \|y\|\leq 1} |\langle Tx, y\rangle|$.*

Proof: With $\varphi_x(y) = \langle y, x\rangle$ we have $\|x\| = \|\varphi_x\| = \sup_{\|y\|\leq 1} |\langle y, x\rangle| = \sup_{\|y\|\leq 1} |\langle x, y\rangle|$. The second assertion follows from the first and the fact that $\|T\| = \sup_{\|x\|\leq 1} \|T(x)\|$. □

Corollary 7 *If $\mathcal{H}$ is a Hilbert space, then the function $y \to \varphi_y$ is an isometric conjugate-linear map of $\mathcal{H}$ onto its dual space $\mathcal{H}^*$. Consequently the dual space of $\mathcal{H}$ is isometrically isomorphic to the Banach space $\mathcal{H}^0$, where $\mathcal{H}^0$ and $\mathcal{H}$ agree as sets and have the same addition and norm, and where the scalar*

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multiplication in $\mathcal{H}^0$ is given by $(\lambda, x) \to \bar{\lambda}x$. The function $[x, y] = \langle y, x\rangle$ is the unique inner product on $\mathcal{H}^0$ which induces the norm of this Banach space. If we put $\langle \varphi_x, \varphi_y\rangle_d = \langle y, x\rangle$, then $\langle\, ,\rangle_d$ is an inner product on $\mathcal{H}^*$ that induces the dual space norm on $\mathcal{H}^*$.

Proof: Exercise. □

Definition 6 The Hilbert space $\mathcal{H}^0$ is called the *opposite Hilbert space* to $\mathcal{H}$.

It is easy to check that the opposite Hilbert space for $\mathcal{H}^0$ is just $\mathcal{H}$. Thus $\mathcal{H}$ and its double dual are naturally isomorphic (and isometric) as Hilbert spaces.

Recall that a Banach space X acts naturally on its dual space X^* as follows: if $x \in X$ and $\phi \in X^*$, then

$$\hat{x}(\phi) = \phi(x)$$

defines a bounded linear map on X^*, since $|\hat{x}(\phi)| \leq \|\phi\|\,\|x\|$. It follows from the Hahn-Banach Theorem (see for example Corollary 1.6.2, p. 44, of [KR]) that the function

$$x \to \hat{x}$$

is an isometric linear map of X into the Banach space double dual X^{**}. This function is called the *canonical injection* of X into X^{**}. If the canonical injection maps X onto X^{**}, then X is said to be a *reflexive* Banach space.

When X is the Hilbert space $\mathcal{H}$, the canonical injection is given by

$$\hat{x}(\varphi_y) = \varphi_y(x) = \langle x, y\rangle = \langle \varphi_y, \varphi_x\rangle_d.$$

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By the Riesz Representation Theorem (applied to the dual space $\mathcal{H}^*$), each bounded linear functional on $\mathcal{H}^*$ has the form $\varphi_y \to \langle \varphi_y, \varphi_x \rangle_d$ for some x in $\mathcal{H}$, and so coincides with $\hat{x}$ for some x in $\mathcal{H}$. Thus the canonical injection of $\mathcal{H}$ maps onto $\mathcal{H}^{**}$, so every Hilbert space is always a reflexive Banach space.

The Riesz Representation Theorem also has an important and useful corollary for sesquilinear forms. A sesquilinear form φ is *bounded* if there exists a non-negative K such that $|\varphi(x, y)| \leq K\|x\| \, \|y\|$ for all x and y. If φ is bounded, then there is a least such K (exercise), which is called the *norm* of φ, and is denoted by $\|\varphi\|$. Thus we have

$$|\varphi(x, y)| \leq \|\varphi\| \, \|x\| \, \|y\| \quad \text{for all } x \text{ and } y.$$

Proposition 7 *If S and T are elements of $\mathcal{B}(\mathcal{H})$, then $S = T$ if and only if $\langle Sx, y \rangle = \langle Tx, y \rangle$ for all x and y if and only if $\langle Sx, x \rangle = \langle Tx, x \rangle$ for all x.*

Proof: For any particular x, $Sx = Tx$ if and only if $Sx - Tx$ is orthogonal to every y in $\mathcal{H}$, so $S = T$ if and only if $\langle Sx, y \rangle = \langle Tx, y \rangle$ for all x and y in $\mathcal{H}$. If we then apply the polarization identity to the sesquilinear forms $\langle Sx, y \rangle$ and $\langle Tx, y \rangle$, we see that $S = T$ if and only if $\langle Sx, x \rangle = \langle Tx, x \rangle$ for all x. □

Theorem 5 *Let φ be a bounded sesquilinear form on a Hilbert space $\mathcal{H}$. Then there exists a unique bounded linear operator T on $\mathcal{H}$ such that for all x and y in $\mathcal{H}$,*

$$\varphi(x, y) = \langle x, T(y) \rangle$$

Moreover the norm of T coincides with the norm of φ.

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Proof: For each y, the linear function $x \to \varphi(x,y)$ on $\mathcal{H}$ is bounded, since $|\varphi(x,y)| \leq \|\varphi\| \, \|x\| \, \|y\|$. Thus there exists a unique z in $\mathcal{H}$ such that $\varphi(x,y) = \langle x, z \rangle$ for all x. Put $T(y) = z$. Then we have

$$\varphi(x,y) = \langle x, T(y) \rangle \quad \text{for all } x \text{ and } y \text{ in } \mathcal{H}.$$

Thus for all x, all y_1 and all y_2, and all scalars λ,

$$\begin{aligned}\langle x, \lambda T(y_1) + T(y_2) \rangle = \bar{\lambda} \langle x, T(y_1) \rangle + \langle x, T(y_2) \rangle &= \bar{\lambda} \varphi(x, y_1) + \varphi(x, y_2) \\ &= \varphi(x, \lambda y_1 + y_2) = \langle x, T(\lambda y_1 + y_2) \rangle,\end{aligned}$$

so T is linear. Now

$$\sup_{\|x\| \leq 1\,, \|y\| \leq 1} |\langle T(y), x \rangle| = \sup_{\|x\| \leq 1\,, \|y\| \leq 1} |\langle x, T(y) \rangle| = \sup_{\|x\| \leq 1\,, \|y\| \leq 1} |\varphi(x,y)| \leq \|\varphi\|,$$

so T is bounded and $\|T\| \leq \|\varphi\|$. But for all x and y,

$$|\varphi(x,y)| = |\langle x, T(y) \rangle| \leq \|x\| \, \|T\| \, \|y\|$$

so $\|\varphi\| \leq \|T\|$. The uniqueness of T is an immediate consequence of the proposition above. □

In the next chapter we shall see that we can also find an operator S in $\mathcal{B}(\mathcal{H})$ with the same norm as φ such that $\varphi(x,y) = \langle S(x), y \rangle$ for all x and y in $\mathcal{H}$.

Exercises

1. Let V be a normed space and W be a Banach space. Show that the space $\mathcal{B}(V, W)$ is a Banach space. (That is, show that $\mathcal{B}(V, W)$ is complete.)

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2. Let T be a bounded linear map from a normed space V to a normed space W. Suppose V is not the zero space.

   a. Show that the number $\|T\| = \inf\{K \geq 0 : \|T(x)\| \leq K\|x\| \text{ for all } x \text{ in } V\}$ satisfies

   $$\|T(x)\| \leq \|T\|\|x\| \quad \text{for all } x \text{ in } V.$$

   b. Show that

   $$\begin{aligned}\|T\| &= \sup\left\{\frac{\|T(x)\|}{\|x\|} : x \neq 0\right\} = \sup\{\|T(x)\| : \|x\| = 1\} \\ &= \sup\{\|T(x)\| : \|x\| \leq 1\}.\end{aligned}$$

3. Prove Corollary 7.

4. Let φ be a bounded sesquilinear form on a Hilbert space. Show that there exists a least non-negative K such that $|\varphi(x, y)| \leq K\|x\|\|y\|$ for all x and y.

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The Adjoint

For each S in $\mathcal{B}(\mathcal{H})$ the function $\varphi(x, y) = \langle S(x), y\rangle$ is a bounded sesquilinear form on $\mathcal{H}$. By the last result of the previous chapter there exists a unique T in $\mathcal{B}(\mathcal{H})$ such that

$$\langle S(x), y\rangle = \varphi(x, y) = \langle x, T(y)\rangle \quad \text{for all } x \text{ and } y \text{ in } \mathcal{H},$$

and $\|T\| = \|\varphi\|$. Taking conjugates to reverse the order of the arguments in the inner product, we see that

$$\sup_{\|x\|\le 1\,,\|y\|\le 1} |\langle T(x), y\rangle| = \sup_{\|x\|\le 1\,,\|y\|\le 1} |\langle y, T(x)\rangle| = \sup_{\|x\|\le 1\,,\|y\|\le 1} |\langle S(y), x\rangle|,$$

That is,

$$\|S\| = \|T\| = \|\varphi\|.$$

Definition 7 Let $S \in \mathcal{B}(\mathcal{H})$. The unique operator T in $\mathcal{B}(\mathcal{H})$ such that $\langle S(x), y\rangle = \langle x, T(y)\rangle$ for all x and y in $\mathcal{H}$ is called the *adjoint* of S and is denoted by S^*.

The adjoint is thus determined by the relation

$$\langle S(x), y\rangle = \langle x, S^*(y)\rangle \quad \text{for all } x \text{ and } y \text{ in } \mathcal{H}.$$

It has the following properties, all of which follow easily from this defining relation (exercise).

Theorem 6 *Let S and T be elements of $\mathcal{B}(\mathcal{H})$, and let λ be a complex number. Then*

1) $(S + T)^* = S^* + T^*$

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2) $(\lambda S)^* = \bar{\lambda} S^*$

3) $(ST)^* = T^* S^*$

4) $S^{**} = S$

5) $\|S\| = \|S^*\|$.

Note that by property 4), the adjoint operation $S \to S^*$ must be a bijection of $\mathcal{B}(\mathcal{H})$ onto itself. The following property of the adjoint is also important, as we shall see when we take up the topic of C^*-algebras.

Proposition 8 *For any bounded T in $\mathcal{B}(\mathcal{H})$, we have $\|T^*T\| = \|T\|^2$.*

Proof: $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2 = \left(\sup_{\|x\|\leq 1} \|Tx\|\right)^2 = \sup_{\|x\|\leq 1} \|Tx\|^2 = \sup_{\|x\|\leq 1}\langle Tx, Tx\rangle = \sup_{\|x\|\leq 1} |\langle Tx, Tx\rangle| \leq \sup_{\|x\|\leq 1, \|y\|\leq 1} |\langle Tx, Ty\rangle| = \sup_{\|x\|\leq 1, \|y\|\leq 1} |\langle T^*Tx, y\rangle| = \|T^*T\|$. □

The following result, though very simple to prove, is extremely important.

Proposition 9 *Let T be an element of $\mathcal{B}(\mathcal{H})$. Then a vector x lies in the kernel of T if and only if x is orthogonal to the range of T^*.*

Proof: We have $Tx = 0$ if and only if $\langle Tx, y\rangle = 0$ for all y in $\mathcal{H}$, so $x \in \text{kernel}(T)$ if and only if $\langle x, T^*y\rangle = 0$ for all y in $\mathcal{H}$. Thus $x \in \text{kernel}(T)$ if and only if x is orthogonal to the range of T^*. □

Corollary 8 *For any T in $\mathcal{B}(\mathcal{H})$,* $\text{kernel}(T)^\perp = \overline{\text{range}(T^*)}$*, and* $\text{range}(T)^\perp = \text{kernel}(T^*)$.

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Proof: From the proposition we have $\text{kernel}(T) = \text{range}(T^*)^\perp$. Thus $\text{kernel}(T)^\perp = \text{range}(T^*)^{\perp\perp} = \overline{\text{range}(T^*)}$. Interchanging the roles of T and T^* in the equation $\text{kernel}(T) = \text{range}(T^*)^\perp$ gives the second assertion. □

The reader should also verify the following: if we allow a square matrix $M = [m_{ij}]$ to operate on $\mathbb{C}^n$ in the usual way, then we get a bounded linear operator on $\mathbb{C}^n$; the adjoint of this operator has matrix equal to the conjugate transpose $[\overline{m_{ji}}]$ of M.

The adjoint operation is of the greatest importance. Three chapters from now, we shall use it to define several of the most useful classes of operators.

Exercises

1. Let M be an n by n matrix with complex entries, and let T be the operator on $\mathbf{C}^n$ given by multiplying a column vector in $\mathbf{C}^n$ by M, with M on the left. Show that the adjoint of T is the operator given by multiplying on the left by the transpose conjugate of M.

2. Let f be a continuous function on $[0, 1]$, and let T be the operator on $L^2[0, 1]$ given by $(Th)(x) = f(x)h(x)$, where x lies in $[0, 1]$. What is the adjoint of T and why?

3. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for a (separable) Hilbert space $\mathcal{H}$. Let S be the unique element of $\mathcal{B}(\mathcal{H})$ that satisfies $S(e_i) = e_{i+1}$ for all i. What is the adjoint of S and why?

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Weak convergence and weak topologies

Let $\mathcal{H}$ be a Hilbert space. Recall that "$\{x_\gamma\}$ converges to x" in $\mathcal{H}$ means that $\|x_\gamma - x\| \to 0$. This kind of convergence is sometimes called "norm convergence" in order to make the connection to the norm explicit. From the Cauchy-Schwarz Inequality we see that if $\{x_\gamma\}$ converges to x, then for each y in $\mathcal{H}$, $\langle x_\gamma, y\rangle \to \langle x, y\rangle$.

Definition 8 We say that $\{x_\gamma\}$ *converges weakly* to x in $\mathcal{H}$ if for every y in $\mathcal{H}$ we have $\langle x_\gamma, y\rangle \to \langle x, y\rangle$.

Thus norm convergence implies weak convergence, and the weak limit of $\{x_\gamma\}$ must coincide with its norm limit if both exist. The next exercise shows in general, weak convergence need not imply norm convergence.

Exercise a) Show that if $\mathcal{H}$ is finite dimensional, then $\{x_\gamma\}$ converges weakly to x in $\mathcal{H}$ if and only if $\{x_\gamma\}$ converges in norm to x in $\mathcal{H}$.
b) Let $\{e_n\}$ be an orthonormal basis for a separable Hilbert space. Show that the sequence $\{e_n\}$ converges weakly to zero, but that this same sequence fails to converge in norm. (To establish weak convergence, consider the Fourier coefficients of y with respect to this basis; to get the fact that $\{e_n\}$ fails to converge in norm, observe that a distance of $\sqrt{2}$ separates any two distinct elements of an orthonormal set.)

Now recall that in any metric space X, a point x lies in the closure of a set S if and only if there exists a sequence $\{x_n\}$ in S such that $\{x_n\}$ converges to x. Thus if we know exactly which sequences converge in X and to what limits, then we know exactly which sets in X are closed, and exactly which

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sets are open. Thus a complete knowledge of which sequences converge to which points in X completely determines the topology of X.

It is tempting to think that weak convergence in $\mathcal{H}$ must also be the convergence associated with some topology on $\mathcal{H}$, and that a complete knowledge of the weak convergence of sequences in $\mathcal{H}$ will determine this topology. This is not quite true, but it does become true if we replace the word "sequences" with "nets." For the reader who is unacquainted with nets, the expectations acquired from working with sequences will lead mostly to correct conclusions, except in a few places that we will try to point out below; a good reference for nets and their relation to topology is G. K. Pederson's *Analysis Now* [PGK]. The topology associated with weak convergence is called the *weak topology.*

Definition 9 The *weak topology* on $\mathcal{H}$ is the weakest topology (in the sense of fewest open sets) that makes all of the linear functionals

$$\varphi_y : x \longrightarrow \langle x, y \rangle, \qquad y \in \mathcal{H}$$

continuous. More generally, the weak topology on a normed space X is the weakest topology which make all of the maps in the Banach space dual X^* of X continuous.

Thus for each vector y in $\mathcal{H}$ and each $\epsilon > 0$, the set $\{z : |\langle z, y \rangle| < \epsilon\}$ is a weakly open neighborhood of zero in $\mathcal{H}$. More generally, for each finite set $\{y_1, y_2, \ldots, y_n\}$ of vectors in $\mathcal{H}$ and each $\epsilon > 0$, $\{z : |\langle z, y_i \rangle - \langle x, y_i \rangle| < \epsilon,$ $i = 1, 2, \ldots, n\}$ is a weakly open neighborhood of x, and the collection of all such sets is a neighborhood basis for the system of weak neighborhoods of x. A net $\{x_\gamma\}$ converges weakly to x (as defined above) if and only if it

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converges to x in the weak topology, and a point x of $\mathcal{H}$ lies in the weak closure of S if and only if there exists a net in S which converges weakly to x. It is not difficult to check (exercise) that the weak topology on $\mathcal{H}$ is Hausdorff. Since each of the linear functionals that define the weak topology is norm continuous, the weak topology is contained in the norm topology. Similar remarks apply to the weak topology on a normed space (which is also Hausdorff), although the action of a linear functional is generally not given by an inner product. It follows from an exercise above that the weak topology on an infinite dimensional $\mathcal{H}$ is strictly weaker than the norm topology.

Exercise Show that if $T \in \mathcal{B}(\mathcal{H})$, then T is continuous from $\mathcal{H}$ with the weak topology to $\mathcal{H}$ with the weak topology.

Proposition 10 *Let $\{x_\gamma\}$ be a net in $\mathcal{H}$, and let $x \in \mathcal{H}$. Then $\|x_\gamma - x\| \to 0$ if and only if $x_\gamma \to x$ weakly and $\|x_\gamma\| \to \|x\|$.*

Proof: $\|x_\gamma - x\|^2 = \langle x_\gamma, x_\gamma\rangle - \langle x_\gamma, x\rangle - \langle x, x_\gamma\rangle + \langle x, x\rangle$. If $x_\gamma \to x$ weakly and $\|x_\gamma\| \to \|x\|$, all four terms tend to $\langle x, x\rangle$, so $\|x_\gamma - x\| \to 0$. If $\|x_\gamma - x\| \to 0$, then $\big|\|x_\gamma\| - \|x\|\big| \to 0$ and $x_\gamma \to x$ weakly. $\square$

Much of the value of the weak topology comes from the fact that it is easier for a subset to be compact in a topology that has fewer open sets. It is a result of Banach that the closed unit ball in a Banach space is norm-compact if and only if the space is finite dimensional. However, it follows from a famous theorem of Alaoglu (see for example [KR, Theorem 1.6.5, p. 45]) that the norm-closed unit ball in a Hilbert space is always weakly compact. Since we shall make considerable use of Alaoglu's Theorem in the sequel, we state it carefully here. To do this we shall need yet another topology, which

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is defined only on a dual space.

Definition 10 Let X be a Banach space and let X^* be its Banach space dual. The *weak* topology* on X^* is the weakest topology which makes all of the linear functionals

$$\varphi \to \varphi(x), \qquad x \in X$$

continuous.

Thus a net $\{\varphi_\gamma\}$ converges weak* to φ in X^* if and only if $\varphi_\gamma(x) \to \varphi(x)$ for all x in X. Since two elements of the dual space are distinct if and only if they differ on some x in X, it is easy to check that the weak* topology is Hausdorff. Note that the functional $\varphi \to \varphi(x)$ is just the image $\hat{x}$ of x under the canonical injection of X into its double dual.

Now the dual space X^* also has a weak topology, defined to be the weakest topology that makes all the functionals in X^{**} continuous. Since the collection $\{\hat{x} : x \in X\}$ is a subset of X^{**}, the weak* topology is always contained in the weak topology on X^*. If X is reflexive, then $\{\hat{x} : x \in X\}$ coincides with X^{**}, so that the weak and weak* topologies on X^* are defined by the same set of linear functionals, and therefore coincide.

Theorem 7 (Alaoglu) *Let X be a Banach space, and let X^* be its Banach space dual. Then the norm-closed unit ball in X^* is weak* compact.*

Proof: See for example [KR, p. 45]. □

Now for any scalar λ and any element ψ of X^*, the functions $\varphi \to \lambda\varphi$ and $\varphi \to \psi + \varphi$ are easily seen to be weak* to weak* continuous from X^* to X^*. The ball $\{\eta \in X^* : \|\eta - \psi\| \leq \lambda\}$ is the image of the norm closed unit ball in X^* under the composition $\eta \to \lambda\eta \to \psi + \lambda\eta$, so it follows that any norm closed ball in a dual space is weak* compact. We recall from topology that a compact subset of a Hausdorff space is always closed [CGL, p. 81], and that a closed subset of a compact space is always compact [CGL, p. 80]. In particular, a weakly compact subset of a Banach space is always weakly closed, any norm closed ball in a dual space is also weak* closed, and we have the following additional consequences of Alaoglu's Theorem.

Corollary 9 *If S is a weak* closed norm-bounded subset of the dual X^* of a Banach space X, then S is weak* compact.*

Proposition 11 *If X is a reflexive Banach space, then the canonical embedding j of X into X^{**} is a homeomorphism of X with the weak toplogy onto X^{**} with the weak* topology.*

Proof: A net $\{x_\gamma\}$ in X converges weakly to x in X if and only if $\varphi(x_\gamma) \to \varphi(x)$ for all $\varphi \in X^*$, i.e., if and only if $[j(x_\gamma)](\varphi) \to [j(x)](\varphi)$ for all $\varphi \in X^*$, i.e., if and only if $\{j(x_\gamma)\}$ converges weak* in X^{**} to $j(x)$. □

Corollary 10 *If X is a reflexive Banach space, then every norm-closed ball in X is weakly compact, and every norm-bounded weakly closed subset of X is weakly compact.*

Proof: Since j is an isometry of X onto X^{**}, the norm-closed ball B in X of radius λ centered at x_0 is the image under j^{-1} of the norm-closed ball B' in X^{**} of radius λ centered at $j(x_0)$. Since B' is weak* compact in X^{**} and j^{-1} is weak*-to-weak continuous from X^{**} to X, it follows that $B = j^{-1}(B')$ is weakly compact in X. If S is norm bounded and weakly closed in X, then S

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is a weakly closed subset of some norm-closed ball in X, so S is itself weakly compact. □

Thus each norm-bounded weakly closed subset of a Hilbert space $\mathcal{H}$ is weakly compact. In particular, any norm-closed ball in $\mathcal{H}$ is weakly closed and weakly compact.

Corollary 11 *Let $\{x_\gamma\}$ be a bounded net in $\mathcal{H}$, and let $x \in \mathcal{H}$. If $x_\gamma \to x$ weakly, then $\|x\| \leq \sup_\gamma \|x_\gamma\|$.*

In order to make effective use of these facts, we shall need the following general results from topology. (See for example [KJL] or [CGL].)

Theorem 8 *A topological space X is compact if and only if every net in X has a convergent subnet.*

Theorem 9 (Tichonoff) *If X is an arbitrary product of compact topological spaces, then X is compact in the product topology.*

Alaoglu's Theorem can be deduced in a straightforward way from the Tichonoff Theorem. (Exercise)

Exercises

1. Show that if $\mathcal{H}$ is finite dimensional, then $\{x_\gamma\}$ converges weakly to x if and only if $\{x_\gamma\}$ converges in norm to x.

2. Let $\{e_n : n = 1, 2, \ldots\}$ be an orthonormal basis for a (separable) Hilbert space $\mathcal{H}$. Show that the sequence $\{e_n\}$ converges weakly to zero, but that $\{e_n\}$ fails to converge in norm. (Consider the Fourier coefficients

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to get the weak convergence; then consider the distance between any two elements of the basis.)

3. Show that the weak topology on $\mathcal{H}$ is Hausdorff. That is, show that for any distinct pair x and y in $\mathcal{H}$ there exist disjoint weakly open neighborhoods of x and y. More generally, use the Hahn-Banach Theorem to show that the weak topology on any Banach space is Hausdorff. (See [KR, Cor. 1.6.2, p. 44].)

4. Show that if T is an element of $\mathcal{B}(\mathcal{H})$, then T is continuous from $\mathcal{H}$ with the weak topology to $\mathcal{H}$ with the weak topology.

5. Deduce Alaoglu's Theorem from the Tichonoff Theorem.

6. Let X be a Banach space.

a) Let $\{\varphi_\gamma\}$ be a net in the dual space X^*, and suppose $\varphi_\gamma \to \varphi$ weak* in X^*. Show that for every γ_0 we have $\|\varphi\| \leq \sup\{\|\varphi_\gamma\| : \gamma_0 \leq \gamma\}$.

b) Suppose X is reflexive, and let $\{x_\gamma\}$ be a net in X. Suppose $x_\gamma \to x$ weakly in X. Show that for every γ_0 we have $\|x\| \leq \sup\{\|x_\gamma\| : \gamma_0 \leq \gamma\}$.

7. Use the Hahn-Banach Theorem (see for example [KR, Thm. 1.2.10, p. 20] to show that a convex subset of a Banach space is norm-closed if and only if it is weakly closed.

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Examples of operators

The identity operator: We shall generally write I for the function defined by $I(x) = x$ for all x in $\mathcal{H}$. This is an isometric linear operator and the identity for the ring multiplication in $\mathcal{B}(\mathcal{H})$.

Matrices on $\mathbb{C}^n$: For each $n \times n$ matrix A, multiplication of column vectors in $\mathbb{C}^n$ by A gives a linear operator in $\mathbb{C}^n$. Since multiplication by A is continuous and the unit ball in $\mathbb{C}^n$ is compact, this multiplication operator is bounded. The multiplication operators defined in the next example below are infinite dimensional analogues of multiplication by diagonal matrices.

Multiplication operators on ℓ^2: Let $c = \{c_n\}$ be a bounded sequence of complex numbers, and let b be a sequence in $\ell^2 = \{\{x_n\} : \sum_n |x_n|^2 < \infty\}$. Then $b = \sum_n b_n e_n$, where e_n is the sequence which is one at n and zero elsewhere. (This sum converges to b in ℓ^2.) Define M_c on ℓ^2 by $M_c(b) = \sum_n c_n b_n e_n$. Since c is bounded, M_c is a bounded operator on ℓ^2, with norm equal to $\|c\|_\infty = \sup_n |c_n|$. If there exists a number $d > 0$ such that $d \leq c_n$ for all n, then M_c has a bounded inverse. Each value c_n is an eigenvalue for M_c, and e_n is an eigenvector for c_n.

If we do not require that c is a bounded sequence, then we can define a multiplication operator M_c on the subspace of ℓ^2 for which the series $\sum_n c_n b_n e_n$ converges in ℓ^2. This operator will not be bounded unless c is bounded.

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Multiplication operators on L^2 : Let f be a measurable function and define M_f on $L^2[0,1]$ by $(M_f h)(t) = f(t)h(t)$. If f is essentially bounded, then M_f is bounded, and in fact $\|M_f\| = \|f\|_\infty$ (the essential supremum of f on $[0,1]$). This operator does not in general have eigenvalues. (Consider for example the function $f(x) = x$.)

Shifts on $\ell^2(N)$: We define left and right shifts by specifying their action on the unit vectors e_n. Put

$$S_R(e_n) = e_{n+1} \quad \text{for all } n \geq 1,$$

and put

$$S_L(e_n) = e_{n-1} \quad \text{for } n > 1 \text{ and } S_L(e_1) = 0.$$

These functions extend uniquely to bounded linear maps S_R and S_L on $\ell^2(N)$, and

$$S_R(\{x_1, x_2, \dots\}) = \{0, x_1, x_2, \dots\} \text{ and } S_L(\{x_1, x_2, \dots\}) = \{x_2, x_3, \dots\}.$$

Both S_R and S_L have norm one, but S_R is one-to one and not onto, while S_L is onto but not one-to-one. Note that $S_L S_R = I$, but $S_R S_L \neq I$, and that S_R is isometric on all of $\ell^2(N)$, while S_L is isometric only on $e_1^\perp = \{x : x_1 = 0\}$.

Shifts on $\ell^2(Z)$: The right shift on $\ell^2(Z)$ is defined as above, except that n ranges through $\mathbf{Z}$ instead of $\mathbf{N}$, while the left shift is given by $S_L(e_n) = e_{n-1}$ for all n. Here S_R and S_L are inverses of one another, and both are bijective and isometric.

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Weighted shifts on $\ell^2(N)$: The weighted shifts are the operators $M_c S_R$ and $M_c S_L$ with M_c, S_R and S_L defined as above.

Integral operators: Let $K(s,t)$ be a function of two variables, defined say, on the unit square $[0,1] \times [0,1]$ in the plane. Assuming $K(s,t)$ satisfies some natural condition, such as continuity or square-integrability, we define T on $L^2[0,1]$ to $L^2[0,1]$ by

$$Tf(t) = \int_0^1 K(s,t)f(s)ds.$$

It is easy to verify that T is bounded (but not so easy to compute the exact value of the norm of T).

Differentiation on $L^2[0,1]$: For some space of differentiable L^2-functions f, define T by

$$Tf = if'.$$

This operator is not defined on all of L^2, but its domain of definition can be taken to be a proper dense subspace of L^2, such as the space of all absolutely continuous L^2-functions f with f' in L^2 and with $f(0) = f(1)$. Note that there is no chance for this operator to be bounded, since small functions can have large rates of change. (Consider for example $\sin(2\pi nx)$ and $\cos(2\pi nx)$.) Note too that there are many other domains that can be used for this operator. Interestingly enough, we shall see below that the properties of unbounded operators like this one depend on the domain as well as the formula of definition.

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Exercises

1. Compute the norms and adjoints of the multiplication operators, shifts and weighted shifts introduced in this chapter.

2. Find bound on the norm of the integral operator $Tf(t) = \int_0^1 K(s,t)f(s)ds$ discussed above.

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Classes of operators

Self-adjoint operators: An element T of $\mathcal{B}(\mathcal{H})$ is *self-adjoint*, or *hermitian*, if $T = T^*$. If T is any element of $\mathcal{B}(\mathcal{H})$, then $T_1 = \frac{T+T^*}{2}$ and $T_2 = \frac{T-T^*}{2i}$ are self-adjoint, and $T = T_1 + iT_2$. It is easy to check that if $T = A + iB$ with A and B self-adjoint, then $A = T_1$ and $B = T_2$. Thus every element of $\mathcal{B}(\mathcal{H})$ has a unique representation in terms of generalized real and imaginary parts, and $\mathcal{B}(\mathcal{H})$ is the span of its self-adjoint elements.

Normal operators: An operator T in $\mathcal{B}(\mathcal{H})$ is normal if it satisfies $TT^* = T^*T$. This is the most general class of bounded linear operators for which we have a fully satisfactory generalization of the process of diagonalizing a matrix with a unitary transformation. The development of this generalization is the central problem of spectral theory.

Exercise Show that T is normal if and only if $\|Tx\| = \|T^*x\|$ for all x in $\mathcal{H}$.

Unitaries: An operator U in $\mathcal{B}(\mathcal{H})$ is *unitary* if it satisfies $UU^* = I = U^*U$.

Thus U is unitary if and only if U is invertible and has U^* for its inverse.

Proposition 12 *For U in $\mathcal{B}(\mathcal{H})$, the following are equivalent:*

a) U is onto and for all x in $\mathcal{H}$, $\|Ux\| = \|x\|$

b) U is onto and for all x and y in $\mathcal{H}$, $\langle Ux, Uy\rangle = \langle x, y\rangle$

c) U is unitary

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Proof: Exercise. □

Thus every unitary is isometric. The bilateral shift is unitary, but the unilateral shifts are not. One of the unilateral shifts gives an example of an isometry that is not unitary. An n by n matrix is unitary if and only if its columns (or rows) form an orthonormal basis for the underlying space $\mathbb{C}^n$. (Exercise.)

Positive Operators: An element T of $\mathcal{B}(\mathcal{H})$ is *positive* if $\langle Tx, x\rangle \geq 0$ for all x in $\mathcal{H}$. Every operator of the form S^*S is clearly positive, and in particular the square of a self-adjoint operator is always positive. Since $\langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} = \langle Tx, x\rangle$ for all x whenever T is positive, any positive operator is necessarily self-adjoint. We shall see below that an operator is positive if and only if it is the square of a self-adjoint operator, and that every self-adjoint operator is the difference of two positive operators whose product is zero. Observe that if M is a square matrix, then multiplication by T is a positive operator if and only if M is positive semidefinite.

Projections: Let V be a vector space and let E be a linear map of V into V. We say that E is a *projection* if $E^2 = E$. Observe that if E is a projection, then so is $I - E$, where I is the identity map. Thus E is a projection if and only if $I - E$ is a projection.

Suppose now that E is a projection. If x lies in the range of E, then for some y in V, $x = Ey$, so that $Ex = E^2y = Ey = x$. It follows that the range of a projection E coincides with the set of fixed points of E. Note that

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$Ex = x$ if and only if $(I - E)(x) = 0$, so that

$$\{x \text{ in } V : Ex = x\} = \text{Range}(E) = \text{Kernel}(I - E).$$

Similarly, $\text{Kernel}(E) = \text{Range}(I - E)$.

Now for each x in V and each projection E, we have $x = Ex + (I - E)(x)$. Since $Ex = (I - E)(y)$ implies $Ex = E^2x = E(I - E)(y) = 0$, it follows that for each projection E, we may write V as a direct sum of the subspaces $\text{Range}(E)$ and $\text{Kernel}(E)$:

$$V = \text{Range}(E) \oplus \text{Kernel}(E).$$

Now suppose that $V = \mathcal{H}$ is a Hilbert space. If $E = E^2$ is bounded, then the kernel of E is closed (it is the inverse image of a point under a continuous map), and the range of E is closed (it is the kernel of $I - E$). An element E of $\mathcal{B}(\mathcal{H})$ is called an *orthogonal projection* if $E^* = E = E^2$, i.e., if E is a self-adjoint projection. Observe that E is an orthogonal projection if and only if $I - E$ is an orthogonal projection, and that every orthogonal projection E is a positive operator (since $E = E^* = E^2$ implies $E = E^*E$).

Proposition 13 *A projection E in $\mathcal{B}(\mathcal{H})$ is orthogonal if and only if the range of E is orthogonal to the kernel of E.*

Proof: Since the range of E is closed, $\text{Range}(E) = \text{kernel}(E^*)^{\perp}$. Thus if $E = E^*$, then the range and kernel of E are orthogonal. Conversely, suppose Range $(E) \perp$ kernel (E). Then for each x,

$$\langle Ex, (I - E)(x)\rangle = 0 = \langle (I - E)(x), Ex\rangle.$$

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But

$$\langle Ex, (I-E)(x)\rangle = 0 \quad \text{if and only if } \langle Ex, x\rangle = \langle Ex, Ex\rangle,$$

and

$$\langle (I-E)(x), Ex\rangle = 0 \quad \text{if and only if } \langle x, Ex\rangle = \langle Ex, Ex\rangle.$$

Thus $\langle Ex, x\rangle = \langle x, Ex\rangle$ holds for all x in $\mathcal{H}$, i.e., $\langle Ex, x\rangle = \langle E^*x, x\rangle$ holds for all x in $\mathcal{H}$. By the polarization identity we have $E = E^*$. □

Example 1 If M is a closed subspace of $\mathcal{H}$, then the orthogonal projection (i.e., the nearest point map) from $\mathcal{H}$ to M has range M and kernel $M^\perp$. It is thus an orthogonal (i.e., self-adjoint) projection in the sense just defined. In fact, every self-adjoint projection E in $\mathcal{B}(\mathcal{H})$ is of this form: take M to be the range of E. Thus the prior use of "orthogonal projection" is entirely consistent with the definition in this section. Note that when E is the orthogonal projection onto M, then $I-E$ is the orthogonal projection onto $M^\perp$, and $\mathcal{H}$ is the Hilbert space direct sum of M and $M^\perp$. (The space $\mathcal{H}$ is the *Hilbert space direct sum* of subspaces M and K if M and K are closed, if $\mathcal{H}$ is the vector space direct sum of M and K, and if M and K are orthogonal in $\mathcal{H}$.)

Proposition 14 *If E is a non-zero orthogonal projection in $\mathcal{B}(\mathcal{H})$, then $\|E\| = 1$.*

Proof: For all x, $\|Ex\|^2 \leq \|Ex\|^2 + \|(I-E)(x)\|^2 = \|x\|^2$, and $\|x\| = \|Ex\|$ whenever x lies in the range of E. □

From this point on, we shall generally abbreviate the term "orthogonal projection" to "projection," since we shall have few occasions to consider any

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other kind. The reader is therefore asked to assume that all projections are bounded, linear and self-adjoint unless named to be otherwise.

Compact operators: A linear map T of a Banach space into itself is said to be *compact* (or *completely continuous*) if the image of the unit ball under T has compact closure (in the norm topology).

It is easy to see that we may use the open unit ball or the closed unit ball, or in fact any ball of strictly positive radius, in this definition without affecting the class of maps that are compact. In fact, T is compact if and only if there exists a ball of strictly positive radius whose image has norm-compact closure, in which case the image of every such ball has norm-compact closure. Since any compact subset of a normed space is always bounded, every compact operator is necessarily a bounded operator. For Hilbert spaces we have a very useful simplification of this definition, which follows from an almost equally useful topological lemma (a distant echo of the Bolzano-Weierstrass Theorem).

Lemma 2 *Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a net in a compact topological space X. Suppose that there exists w in X such that every convergent subnet of $\{x_\gamma\}_{\gamma\in\Gamma}$ converges to w. Then $\{x_\gamma\}_{\gamma\in\Gamma}$ converges.*

Proof: Suppose $x = \{x_\gamma\}_{\gamma\in\Gamma}$ does not converge. Then in particular it fails to converge to w. Thus there exists an open neighborhood N of w with the property that $\Gamma_N = \{\gamma \in \Gamma : x_\gamma \notin N\}$ is cofinal in Γ. (That is, for each $\eta \in \Gamma$ there exists $\gamma_\eta \in \Gamma_N$ such that $\eta \leq \gamma_\eta$.) Since the inclusion map of Γ_N into Γ is monotone, $x_N = \{x_\gamma\}_{\gamma\in\Gamma_N}$ is a subnet of x, and this subnet x_N

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takes all its values in the complement C of N. Since X is compact, x_N has a convergent subnet z, and z is also a subnet of x. Thus (by hypothesis) z converges to w. But the net z also takes its values C, and C is closed. It follows that the limit w must lie in C, which contradicts the fact that the complement N of C is a neighborhood of w. $\square$

Proposition 15 *Let B denote the closed unit ball of a Hilbert space $\mathcal{H}$. Let T be a linear map of $\mathcal{H}$ into $\mathcal{H}$. Then the following are equivalent.*

1) T is compact;

2) The image $T(B)$ of B is norm-compact;

3) T is weak to norm continuous from B to $\mathcal{H}$.

Proof: 1) implies 2): Suppose T is compact. Then T is bounded, so T is weak to weak continuous on $\mathcal{H}$, so $T(B)$ is weakly compact. The weak topology is Hausdorff, so $T(B)$ is weakly closed. Now the norm topology contains the weak topology, so $T(B)$ is norm-closed. Since T is compact, $T(B)$ has norm-compact closure, which must coincide with $T(B)$. Thus $T(B)$ is norm-compact.

2) implies 3): Suppose that $T(B)$ is norm-compact; we must show that T is weak to norm continuous from B to $\mathcal{H}$. To do this, let $x_\gamma \to x$ weakly in B, and it suffices to show that $Tx_\gamma \to Tx$ in norm. Since $T(B)$ is compact, $T(B)$ is bounded, so T is a bounded operator. Thus T is weak to weak continuous, so $Tx_\gamma \to Tx$ weakly in $T(B)$. It follows that every subnet of $\{Tx_\gamma\}$ converges weakly to Tx, and hence that every norm convergent subnet of $\{Tx_\gamma\}$ must converge (in norm) to Tx. By the lemma, $\{Tx_\gamma\}$

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must converge in norm. Since $\{Tx_\gamma\}$ converges weakly to Tx, $\{Tx_\gamma\}$ must converge in norm to Tx, and the implication 2) $\Rightarrow$ 3) is established.

3) implies 1): Suppose T is weak to norm continuous from B to $\mathcal{H}$. Then $T(B)$ is norm-compact, since B is weakly compact. In particular, T is compact. □

Corollary 12 *If T is compact, then T is weak to norm continuous on bounded subsets of $\mathcal{H}$.*

Proof: For any $\lambda > 0$, multiplication by λ is a weak to weak homeomorphism of $\mathcal{H}$ to itself. □

Remark 5 If T is compact, then $T(B)$ is compact in the norm topology and Hausdorff in the weak topology. Thus every norm-closed subset of $T(B)$ is norm-compact, and every weakly compact subset of $T(B)$ is weakly closed. It follows that the identity map on $T(B)$ takes norm closed sets to weakly closed sets. The identity map is thus continuous on $T(B)$ from the weak to the norm topology. Since it is trivially continuous from the norm to the weak, these two topologies coincide on $T(B)$ whenever T is compact. If T is bounded, and if the weak and norm topologies coincide on $T(B)$, then since $T(B)$ is weakly compact, $T(B)$ is also norm compact, so T is compact. Thus a bounded linear T is compact if and only if the weak and norm topologies coincide on $T(B)$.

If T is bounded and the range of T is finite-dimensional, then T is compact, since any closed bounded subset of a finite dimensional normed space is always compact. However, in any infinite dimensional Hilbert space $\mathcal{H}$,

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any two orthogonal unit vectors are separated by a distance of $\sqrt{2}$, so any orthonormal basis for $\mathcal{H}$ is a norm-closed subset of $\mathcal{H}$. If the closed unit ball in $\mathcal{H}$ were compact, then this basis (a closed subset of the unit ball) would also be compact, which would contradict the fact that the set of all open balls of radius $\sqrt{2}/2$ centered at the points of the basis is an open cover of the basis without a finite subcover. Thus the closed unit ball in an infinite dimensional $\mathcal{H}$ is not a compact set, so the identity map is not a compact operator on $\mathcal{H}$.

We shall take up the compact operators systematically later, and we shall show that every normal compact operator is diagonalizable in a way that is very much like the unitary diagonalization of a normal matrix. For now we mention only the following result.

Theorem 10 *Let T and S be linear maps of $\mathcal{H}$ into $\mathcal{H}$. Then:*

1) *If T is compact, then $T \in \mathcal{B}(\mathcal{H})$.*

2) *If $\{T_n\}$ is a sequence of compact operators, T is bounded and $\|T_n - T\| \to 0$, then T is compact.*

3) *If the range of T is finite dimensional and T is bounded (we say then that T is a finite rank operator), then T is compact.*

4) *If $E \in \mathcal{B}(\mathcal{H})$ is a self-adjoint projection, then E is compact if and only if it has finite dimensional range.*

5) *If T is bounded, then T is compact if and only if T^* is compact.*

6) *If T and S are compact and λ is a scalar, then $\lambda T + S$ is compact.*

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7) If T is compact and S is bounded, then both TS and ST are compact.

Proof: 1) and 3) are established in remarks above. If the range of a bounded self-adjoint projection E is infinite dimensional, then choose an infinite orthonormal system in the range; an argument similar to the one used above to show that the identity is not compact will show that E is not compact. If E has finite dimensional range, then it is compact by 3). Thus 4) is established. 6) follows from the fact that a sum of compact sets is compact.

To prove 2), suppose $\|T_n - T\| \to 0$ with each T_n compact. Consider $\{Tx_\gamma\}$, where $\{x_\gamma\}$ is a net in the closed unit ball B. We claim that $\{Tx_\gamma\}$ has a norm convergent subnet. Passing to a subnet of $\{x_\gamma\}$, we may assume that $x_\gamma \to x$ weakly for some x in B. Then $\lim_\gamma \|T_n x_\gamma - T_n x\| \to 0$, since compact operators are weak to norm continuous on B. Now choose N so that $n \geq N$ implies $\|T_n - T\| < \frac{\epsilon}{4}$. Then for $n \geq N$ we have

$$\begin{aligned}\|Tx_\gamma - Tx\| &\leq \|(T-T_n)(x_\gamma - x)\| + \|T_n(x_\gamma - x)\| \\ &\leq \|(T-T_n)(x_\gamma)\| + \|(T-T_n)(x)\| + \|T_n(x_\gamma - x)\| \\ &\leq \frac{\epsilon}{2} + \|T_n(x_\gamma - x)\|.\end{aligned}$$

Since $x_\gamma \to x$ weakly and T_n is compact, $\|T_n(x_\gamma - x)\| \to 0$ for each $n \geq N$. It follows that $\{Tx_\gamma\}$ converges to Tx in norm. Thus the original net $\{Tx_\gamma\}$ in $T(B)$ has a norm convergent subnet, so $T(B)$ is norm-compact, so T is compact.

To prove 5), suppose T is compact. Let $x_\gamma \to x$ weakly in B. It suffices to show that $T^*x_\gamma \to T^*x$ in norm. Since T^* is weakly continuous, $T^*x_\gamma \to T^*x$ weakly. Thus to prove that $\|T^*x_\gamma - T^*x\| \to 0$ it suffices to show that $\|T^*x_\gamma\| \to \|T^*x\|$, i.e., that $\langle T^*x_\gamma, T^*x_\gamma\rangle \to \langle T^*x, T^*x\rangle$, i.e., that

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$\langle x_\gamma, TT^*x_\gamma\rangle \to \langle x, TT^*x\rangle$. Now T is weak to norm continuous on bounded sets, since T is compact, and $\sup_\gamma \|T^*x_\gamma\| \leq \|T^*\|$, since each x_γ lies in B. Thus $\|TT^*x_\gamma - TT^*x\| \to 0$. Now

$$\begin{aligned}|\langle x_\gamma, TT^*x_\gamma\rangle - \langle x, TT^*x\rangle| &\leq |\langle x_\gamma, TT^*x_\gamma\rangle - \langle x_\gamma, TT^*x\rangle| \\ &\quad + |\langle x_\gamma, TT^*x\rangle - \langle x, TT^*x\rangle| \\ &\leq \|x_\gamma\|\|TT^*x_\gamma - TT^*x\| + |\langle x_\gamma - x, TT^*x\rangle| \\ &\leq \|TT^*x_\gamma - TT^*x\| + |\langle x_\gamma - x, TT^*x\rangle|.\end{aligned}$$

Since $x_\gamma \to x$ weakly and $\|TT^*x_\gamma - TT^*x\| \to 0$, $\langle x_\gamma, TT^*x_\gamma\rangle \to \langle x, TT^*x\rangle$ as desired.

To prove 7), observe that if T is compact, then $T(B)$ is norm-compact. If also S is bounded and linear, then $S(T(B))$ is a norm-continuous image of a norm-compact set, so ST is compact. Also, T^* is compact (by 5)), so S^*T^* is compact, so $TS = (S^*T^*)^*$ is compact. □

In a language that we shall develop below, this theorem asserts that the set of all compact operators on $\mathcal{H}$ is a closed adjoint-stable two-sided ideal in the algebra $\mathcal{B}(\mathcal{H})$, and that this ideal contains all bounded finite rank operators.

Exercises

1. Show that T is normal if and only if $\|Tx\| = \|T^*x\|$ for all x in $\mathcal{H}$. Show also that if T is normal, then ker T = ker T^* and $\overline{R(T)} = \overline{R(T^*)}$. (We shall see in Part II that if T is normal, then in fact $R(T) = R(T^*)$.)

2. Prove Proposition 12.

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3. Verify that multiplication by an n by n complex matrix is a unitary operator if and only if the columns (or rows) of the matrix form an orthonormal basis for $\mathbf{C}^n$.

4. Show that if E is an orthogonal projection in $\mathcal{B}(\mathcal{H})$, then $I - 2E$ is a self-adjoint unitary. Give a description of $I-2E$. Also give an example of a non-self-adjoint projection in $\mathcal{B}(\mathcal{H})$.

5. Let T be linear from a Hilbert space $\mathcal{H}$ to itself, and let B and O denote respectively the closed and open unit ball in $\mathcal{H}$. Show that the norm-closure of $T(B)$ is norm-compact if and only if the norm-closure of $T(O)$ is norm-compact. Then show that if U is any ball (open or closed, not necessarily centered at zero) of strictly positive radius in $\mathcal{H}$, then T is compact if and only if the norm-closure of $T(U)$ is norm-compact.

6. Let $K(s,t)$ be a continuous function on the unit square $[0,1] \times [0,1]$ in the plane. Show that the integral operator Tf, defined by $Tf(t) = \int_0^1 K(s,t)f(s)ds$, is compact.

7. When is a multiplication operator on $\ell^2(N)$ compact?

8. When is a multiplication operator on $L^2[0,1]$ compact?

9. Let T be the weighted shift on $\ell^2(N)$ given by $Tx = \sum_{n=1}^{+\infty} c_n \langle x, e_n \rangle e_{n+1}$. Show that if $c_n \to 0$, then T is compact.

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Invertibility and the spectrum of an operator

The spectrum of an operator is a generalization of the set of eigenvalues of a square matrix. To see how this generalization should be made, consider a linear map T of a finite dimensional space V into itself.

An eigenvalue of T is a number λ such that for some non-zero vector v (called an eigenvector for λ) we have $Tv = \lambda v$, i.e., $(T - \lambda I)v = 0$. Thus λ is an eigenvalue of T if and only if $T - \lambda I$ has a non-zero kernel. The rank and nullity theorem tells us that for any linear map S of V into V, the dimension of V is the sum of the dimension of the range of S and the dimension of the kernel of S; thus S is one-to-one if and only if S is onto. It follows immediately that λ is an eigenvalue of T if and only if $T - \lambda I$ *fails* to map onto V. Thus for finite dimensional V, λ is an eigenvalue of T if and only if any one of the following conditions holds (in which case they all hold):

1) $T - \lambda I$ fails to be one-to-one;

2) $T - \lambda I$ fails to be onto;

3) $T - \lambda I$ is not invertible.

As we can see from examples of operators on $\ell^2(N)$, the equivalence of "S is one-to-one" and "S is onto" no longer holds in infinite dimensions: the left shift is onto but not one-to-one, while the right shift is one-to-one but not onto. Moreover, a bounded linear T can fail to be invertible in $\mathcal{B}(\mathcal{H})$, yet have no eigenvectors for $\lambda = 0$, as the next example shows.

Example 2 Let M be multiplication on $L^2[0,1]$ by $f(x) = x$. Let h be the characteristic function of a subinterval (a, b) of $[0, 1]$. Any bounded inverse

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W for M must satisfy $WMh = h$ almost everywhere on $[0,1]$, so that $\|W\| \geq \frac{\|h\|}{\|Mh\|}$. Now $\|h\|^2 = \int_0^1 h^2 = \int_a^b 1 = b-a$, while $\|Mh\|^2 = \int_a^b x^2 dx = \frac{1}{3}(b^3-a^3)$, so $\|W\|^2 \geq \frac{3}{b^2+ab+a^2}$. By choosing a and b very close to zero, we see that the norm of W would need to be arbitrarily large, so M has no inverse in $\mathcal{B}(L^2[0,1])$. On the other hand, if $xg(x) = 0$ almost everywhere on $[0,1]$, then clearly $g(x) = 0$ almost everywhere, so $g = 0$ in $L^2[0,1]$. Thus M is one-to-one on $L^2[0,1]$, so M has no eigenvectors for $\lambda = 0$.

To sort all this out, we make the following definitions.

Definition 11 Let T be an element of $\mathcal{B}(\mathcal{H})$. The *spectrum* of T, denoted by $\mathrm{sp}(T)$ or $\sigma(T)$, is the set of all complex numbers λ such that $T - \lambda I$ fails to have an inverse in $\mathcal{B}(\mathcal{H})$. A number λ is an *eigenvalue* for T if $T - \lambda I$ fails to be one-to-one. If λ is an eigenvalue for T, a non-zero element of the kernel of $T - \lambda I$ is called an *eigenvector* of T, and the kernel of $T - \lambda I$ is called the *eigenspace* of T corresponding to λ.

Thus the eigenvalues of T, if they exist, form a part of the spectrum of T.

Exercise a) Show that multiplication by $f(x) = x$ on $L^2[0,1]$ has spectrum equal to $[0,1]$, and that no element of this spectrum is an eigenvalue. b) Let x be a sequence which converges to zero, and let M be multiplication by the sequence x on ℓ^2. Show that the spectrum of M is the closure of the range of x (i.e., the union of the range of x and $\{0\}$), that every element of the range of x is an eigenvalue of M, and that zero is not an eigenvalue of M unless zero lies in the range of x.

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Clearly we need some criteria for invertibility in $\mathcal{B}(\mathcal{H})$.

Theorem 11 *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

a) T has a bounded inverse;

b) T^* has a bounded inverse;

c) There is a constant $k > 0$ such that $\|Tx\| \geq k\|x\|$ and $\|T^*x\| \geq k\|x\|$ for every x in $\mathcal{H}$;

d) T and T^* are one-to-one and the range of T is closed;

e) T is one-to-one and onto.

Lemma 3 *Suppose there exists a constant $k > 0$ such that $\|Tx\| \geq k\|x\|$ for all x in $\mathcal{H}$. Then T is one-to-one and the range of T is closed.*

Proof: Clearly T is one-to one. Suppose $y_n = Tx_n$ is a sequence in the range of T, and suppose that $\{y_n\}$ converges to some y in $\mathcal{H}$. Then $\|y_m - y_n\| \geq k\|x_m - x_n\|$ and $k > 0$ imply that $\{x_n\}$ is a Cauchy sequence. Thus $\{x_n\}$ converges to some x in $\mathcal{H}$. Since T is continuous, y_n converges to Tx, so $y = Tx$ lies in the range of T. Thus T has closed range. □

Proof of Theorem 11: Since $(AB)^* = B^*A^*$ for all A and B and $I^* = I$, it is clear that a) is equivalent to b). Suppose a) and b) hold. Then $\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\|$ for all x, so $\frac{1}{\|T^{-1}\|}\|x\| \leq \|Tx\|$ for all x; similarly $\frac{1}{\|(T^*)^{-1}\|}\|x\| \leq \|T^*x\|$ for all x. Thus we may take the constant k in part c) to be the minimum of $\frac{1}{\|T^{-1}\|}$ and $\frac{1}{\|(T^*)^{-1}\|}$, and c) is established. The lemma shows that c) implies d). For d) implies e), we need only show that the range of T is all of $\mathcal{H}$. By d), the kernel of T^* is $\{0\}$, so that the closure of the

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range of T is $\mathcal{H}$. Since the range of T is closed, e) is established. Finally e) implies a) by the open mapping theorem (or the closed graph theorem). $\square$

Remark 6 Since $S(T - \lambda I) = I = (T - \lambda I)S$ if and only if $(T^* - \bar{\lambda} I)S^* = I = S^*(T^* - \bar{\lambda} I)$, we see that λ lies in the spectrum of T if and only if $\bar{\lambda}$ lies in the spectrum of T^*.

We shall shortly see that the set of invertible elements of $\mathcal{B}(\mathcal{H})$ is an open subset of $\mathcal{B}(\mathcal{H})$, and that the spectrum of any element of $\mathcal{B}(\mathcal{H})$ is always a non-empty compact subset of $\mathbb{C}$. These facts are no more difficult to establish in a more general setting, so we shall defer the proofs until we take up the subject of Banach algebras. We conclude this section with a classification of the elements λ of the spectrum according to the ways in which $T - \lambda I$ fails to be invertible.

The *point spectrum* of T, denoted by $\sigma_p(T)$, is the set of all eigenvalues of T, i.e., the set of all λ for which $T - \lambda I$ is not one-to-one. The *continuous spectrum* $\sigma_c(T)$ is the set of all λ for which $T - \lambda I$ is one-to-one and the range of $T - \lambda I$ is a proper dense subspace of $\mathcal{H}$. The *residual spectrum* is the remaining part, i.e., is the set of all λ for which $T - \lambda I$ is one-to-one and the closure of the range of $T - \lambda I$ is a proper subspace of $\mathcal{H}$. The spectrum of each T is the disjoint union of its point spectrum, its continuous spectrum and its residual spectrum. The *approximate point spectrum*, $\sigma_{ap}(T)$ consists of those points which are eigenvalues or nearly eigenvalues of T : $\sigma_{ap}(T)$ consists of all those λ in the spectrum such that for every $\epsilon > 0$ there is a unit vector x in $\mathcal{H}$ with $\|(T - \lambda I)x\| < \epsilon$.

Proposition 16 *For any T in $\mathcal{B}(\mathcal{H})$, $\sigma_c(T) \subseteq \sigma_{ap}(T)$.*

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Proof: If λ is an element of $\sigma(T)$ but λ is not an element of $\sigma_{ap}(T)$, then there exists $k > 0$ such that $\|Tx - \lambda x\| \geq k\|x\|$ for all x in $\mathcal{H}$. By the lemma above, the range of $T - \lambda I$ is closed and $T - \lambda I$ is one-to-one. Since λ is an element of $\sigma(T)$ and $T - \lambda I$ is one-to-one, $T - \lambda I$ is not onto. Since its range is closed, its range is a proper closed subspace of $\mathcal{H}$, so is not dense. Thus λ lies in the residual spectrum. In particular, λ is not in the continuous spectrum. □

Proposition 17 *If T is a bounded normal operator, then the residual spectrum of T is empty.*

Corollary 13 *If T is a bounded normal operator, then the spectrum of T agrees with the approximate point spectrum of T.*

Proof: Suppose λ is an element of the residual spectrum of T, so that the range of $T - \lambda I$ is not dense in $\mathcal{H}$. Then we have kernel$(T^* - \bar{\lambda} I) =$ Range$(T - \lambda I)^{\perp} \neq \{0\}$. Since T is normal, $T - \lambda I$ is normal. Thus $\|(T^* - \bar{\lambda} I)x\| = \|(T - \lambda I)x\|$ for all x, so $\ker(T^* - \bar{\lambda} I) = \ker(T - \lambda I)$. Thus the kernel of $T - \lambda I$ is non-zero, so λ lies in the point spectrum of T. But this contradicts the fact that the point spectrum and the residual spectrum have empty intersection. Thus the residual spectrum must be empty. □

Examples (Exercise):

The spectrum of the identity operator I is $\{1\}$, and this coincides with the point spectrum. Every eigenvalue of a matrix lies in the point spectrum of the corresponding operator, and the spectrum of this operator is precisely the set of eigenvalues.

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If T is multiplication by the function $f(x) = x$ on $L^2[0,1]$, then every element of $[0,1]$ lies in the continuous spectrum of T. (In particular, this implies that the elements of $L^2[0,1]$ of the form $xh(x)$, where h lies in $\mathcal{H}$, form a dense proper subspace of $\mathcal{H}$; this can be established by using the Weierstrass approximation theorem.) Moreover the spectrum of this T is precisely $[0,1]$.

For each t with $|t| < 1$, the geometric sequence $\{1, t, t^2, \dots\}$ is an eigenvector with eigenvalue t for the left shift $S = S_L$. For each such t, $(S - tI)^*$ is one-to-one; the range of $(S - tI)^*$ is contained in the orthogonal complement of $\{1, t, t^2, \dots\}$, so the conjugate of each such t lies in the residual spectrum of $S^* = S_R$.

Exercises

1. Show that multiplication by $f(x) = x$ on $L^2[0,1]$ has spectrum equal to $[0,1]$, and that no element of this spectrum is an eigenvalue of this multiplication operator. What is the approximate point spectrum of this operator, and why?

2. Let T be multiplication by the function $f(x) = x$ on $L^2[0,1]$. Show that every element of $[0,1]$ lies in the continuous spectrum of T. Use the Weierstrass approximation theorem to show directly that the range of T is a dense proper subspace of $\mathcal{H}$.

3. Let x be a sequence which converges to zero, and let M be multiplication by the sequence x on l^2. Show that the spectrum of M is the closure of the range of x (i.e., the union of the range of x and $\{0\}$),

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that every element of the range of x is an eigenvalue of M, and that zero is not an eigenvalue of M unless zero lies in the range of x.

4. Show that the spectrum of the identity operator I is $\{1\}$, and that this spectrum coincides with the point spectrum. Show also that every eigenvalue of a square matrix lies in the point spectrum of the corresponding operator, and the spectrum of this operator is precisely the set of eigenvalues.

5. Show that for each complex t with $|t| < 1$, the geometric sequence $\{1, t, t^2, \ldots\}$ is an eigenvector with eigenvalue t for the left shift $S = S_L$ on $l^2(\mathbb{N})$. Then show that each complex t with $|t| < 1$ lies also in the residual spectrum of $S^* = S_R$.

6. What is the spectrum of a projection in $\mathcal{B}(\mathcal{H})$? Does it matter if the projection is orthogonal?

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Banach algebras and the nature of the spectrum

Our next goal is to show that the spectrum of any bounded linear operator on a Hilbert space $\mathcal{H}$ is a non-empty compact subset of the plane. This result is actually true much more generally.

Definition 12 An *algebra* over a field F is a set A which is both a ring and a vector space over F, and in which the vector space and ring operations obey the law

$$\lambda(ab) = a(\lambda b) = (\lambda a)b$$

for all scalars λ and all elements a and b of A. An *ideal* in an algebra is a vector subspace which is also an ideal in the underlying ring structure. A *subalgebra* is a vector space which is also a subring. An *identity* for an algebra A is a non-zero element I of A such that $Ia = a = aI$ for all a in A. An element a of an algebra with an identity I is *invertible* if there exists b in A with $ab = I = ba$. An *algebra homomorphism* is a map that is both linear and a ring homomorphism; an *algebra isomorphism* is a bijective algebra homomorphism.

Every ideal in an algebra A is evidently a subalgebra of A.

Examples Any field is an algebra over any subfield of itself. (Use the field multiplication for both the ring multiplication and the multiplication by a scalar.) The n by n matrices over a field F form an algebra over F. $\mathcal{B}(\mathcal{H})$ is an algebra over $\mathbb{C}$ (or $\mathbb{R}$). The space of all compact operators on $\mathcal{H}$ is an ideal in $\mathcal{B}(\mathcal{H})$. The quaternions form an algebra over $\mathbb{R}$ but not over $\mathbb{C}$. (Exercise:

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Why not?). If X is a set, then the space $\mathbb{C}^X$ of all functions from X into $\mathbb{C}$, provided with pointwise operations, is an algebra over $\mathbb{C}$. The space $C(X)$ of all continuous functions from a topological space X into $\mathbb{C}$ is an algebra over $\mathbb{C}$ and a subalgebra of $\mathbb{C}^X$ (but not an ideal in $\mathbb{C}^X$). The space $C_c(\mathbb{R})$ of all continuous functions from $\mathbb{R}$ into $\mathbb{C}$ that have compact support is an ideal in $C(\mathbb{R})$.

All of the algebras in the previous paragraph have identities except for the algebra of all compact operators on $\mathcal{H}$ and the algebra $C_c(\mathbb{R})$. (Note that an ideal in an algebra A contains an identity of A if and only if it is all of A.) Another example of an algebra without an identity is the algebra $C_0(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ that vanish at infinity. (A function f on a topological space X *vanishes at infinity* if for every $\epsilon > 0$ there exists a compact subset K of X such that for all x in the complement of K we have $|f(x)| \leq \epsilon$.) Observe that $C_c(\mathbb{R})$ is an ideal in $C_0(\mathbb{R})$ and that $C_0(\mathbb{R})$ is an ideal in $C(\mathbb{R})$.

Definition 13 A *normed algebra* is an algebra A over $\mathbb{C}$ which is also a normed vector space, and in which we have

$$\|ab\| \leq \|a\|\|b\|$$

for all elements a and b of A. A complete normed algebra is called a *Banach algebra*.

Examples $\mathcal{B}(\mathcal{H})$, the space of all compact operators on $\mathcal{H}$, $C_0(\mathbb{R})$, and $C(X)$ are all Banach algebras under the operator norm or the supremum norm. $L^\infty[0, 1]$ is a Banach algebra under the essential supremum norm.

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$C_c(\mathbb{R})$ and the space of all finite rank operators on $\mathcal{H}$ are normed algebras; $C_c(\mathbb{R})$ is not complete, and the algebra of all finite rank operators is complete only when $\mathcal{H}$ is finite dimensional.

Let A be a normed algebra with identity I. Then $\|I\| = \|I^2\| \leq \|I\|^2$. Since I is not zero, we must have $1 \leq \|I\|$. We shall always assume that in any normed algebra with identity, the identity has norm one. This is not much of a restriction, since the algebra can always be given an equivalent norm in which this is true. (See the exercises.)

Definition 14 Let A be an algebra with identity over a field F, and let a be an element of A. The *spectrum* of a with respect to A is the set

$$\mathrm{sp}(a) = \mathrm{sp}_A(a) = \{\lambda \text{ in } F : a - \lambda I \text{ is not invertible in } A\}.$$

Clearly this notion generalizes that given in a previous section for the spectrum of an element of $\mathcal{B}(\mathcal{H})$. Note that the spectrum as here defined depends on A as well as on a. Indeed, suppose $n > k$, and suppose we embed the algebra $M_k(\mathbb{C})$ of all k by k matrices over $\mathbb{C}$ into the algebra $M_n(\mathbb{C})$ of all n by n matrices over C by writing each k by k matrix in the upper left hand corner of an n by n matrix and filling out the larger matrix with zeros. This gives a subalgebra B of $M_n(\mathbb{C})$ that is isomorphic to $M_k(\mathbb{C})$, but every element of B is singular in $M_n(\mathbb{C})$. Thus if b is an invertible element of the algebra B, then $\mathrm{sp}_B(b)$ does not coincide with the spectrum of b in $M_n(\mathbb{C})$.

Even if A and B have the same identity, it need not happen that $\mathrm{sp}_B(b) = \mathrm{sp}_A(b)$. However, if B is a subalgebra of A, and if the identity of A lies in B, then for each b in B, $\mathrm{sp}_B(b) \supseteq \mathrm{sp}_A(b)$. (Anything which is invertible in the

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smaller algebra is invertible in the larger, with the same inverse.) Note that the spectrum is an isomorphism invariant: if T is an algebra isomorphism of A onto B, then for each a in A the spectrum of $T(a)$ in B agrees with that of a in A.

In a normed algebra, the multiplication is jointly continuous.

Proposition 18 *Let A be a normed algebra. If $a_n \to a$ and $b_n \to b$ in A, then $a_n b_n \to ab$.*

Proof: Let $a_n \to a$ and $b_n \to b$ in A. Then

$$\begin{aligned}
\|a_n b_n - ab\| &\leq \|a_n b_n - a_n b\| + \|a_n b - ab\| \\
&= \|a_n(b_n - b)\| + \|(a_n - a)b\| \\
&\leq \|a_n\| \|(b_n - b)\| + \|(a_n - a)\| \|b\|.
\end{aligned}$$

Since $a_n \to a$, the second term $\|(a_n - a)\| \|b\|$ tends to zero. Also since $a_n \to a$, $\|a_n\|$ is bounded, so the first term $\|a_n\| \|(b_n - b)\|$ tends to zero. □

The next proposition shows that in a Banach algebra A with identity I, I is an interior point of the set of invertible elements of A. (We need completeness of A for this result.)

Proposition 19 *If A is a Banach algebra with identity I, and if $\|a\| < 1$, then $I - a$ is invertible, and $\|(I - a)^{-1}\| \leq \frac{1}{1 - \|a\|}$.*

Proof: Consider the series $\sum_{k=0}^{+\infty} a^k$. Since $\|a^k\| \leq \|a\|^k$ for all k, we have for all n and m that $\|\sum_{k=n}^{m} a^k\| \leq \sum_{k=n}^{m} \|a^k\| \leq \sum_{k=n}^{m} \|a\|^k$. Since $\|a\| < 1$, $\sum_{k=0}^{+\infty} \|a\|^k$ is a convergent geometric series, and it follows that the partial sums of $\sum_{k=0}^{+\infty} a^k$ are Cauchy in A. Since A is complete, this series converges in

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A. Now for each n we have $(I-a)\left(\sum_{k=0}^{n} a^k\right) = I - a^{n+1} = \left(\sum_{k=0}^{n} a^k\right)(I-a)$, and $\|a^{n+1}\| \leq \|a\|^{n+1} \to 0$ as $n \to +\infty$. It follows from the continuity of multiplication in A that $(I-a)\left(\sum_{k=0}^{+\infty} a^k\right) = I = \left(\sum_{k=0}^{+\infty} a^k\right)(I-a)$, so that $\sum_{k=0}^{+\infty} a^k$ is an inverse for $I-a$. Now for all n we have $\|\sum_{k=0}^{n} a^k\| \leq \sum_{k=0}^{n} \|a\|^k \leq \sum_{k=0}^{+\infty} \|a\|^k = \frac{1}{1-\|a\|}$, so $\|(I-a)^{-1}\| = \|\sum_{k=0}^{+\infty} a^k\| \leq \frac{1}{1-\|a\|}$. □

Corollary 14 *If A is a Banach algebra with identity I, if a is an element of A, and if $\|I-a\| < 1$, then a is invertible, and $\|a^{-1}\| \leq \frac{1}{1-\|I-a\|}$.*

Proof: Apply the previous result to $b = I - a$. □

Proposition 20 *Let A be a Banach algebra with identity I, and let a be an invertible element of A. If $\|b-a\| < \|a^{-1}\|^{-1}$, then b is invertible.*

Proof: Suppose $\|b-a\| < \|a^{-1}\|^{-1}$. Then $\|a^{-1}(a-b)\| \leq \|a^{-1}\|\|a-b\| < 1$, so $I - a^{-1}(a-b)$ is invertible. But $b = a - (a-b) = a[I - a^{-1}(a-b)]$, so b is the product of two invertible elements, and hence is invertible. □

Corollary 15 *The set of all invertible elements of A is an open subset of A.*

Proposition 21 *Let A be a Banach algebra with identity I. Let $a_n \to a$ in A, and suppose a is invertible. Then for all sufficiently large n, a_n is invertible, and $(a_n)^{-1} \to a^{-1}$. That is, inversion is continuous on the set of invertible elements of A.*

Proof: Since the invertible elements of A form an open set, it is clear that for all sufficiently large n, a_n is invertible. Thus we may assume that a_n is invertible for all n. By the continuity of multiplication, we have $a_n \to a$ if and only if $a^{-1}a_n \to I$, and $(a_n)^{-1} \to a^{-1}$ if and only if $(a_n)^{-1}a \to I$, so we may

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assume that $a = I$. Now since $\|a_n^{-1} - I\| = \|a_n^{-1}(I - a_n)\| \leq \|a_n^{-1}\|\|I - a_n\|$, it suffices to show that $\|a_n^{-1}\|$ is bounded. But if $\|a_n - I\| < \frac{1}{2}$, then $\|a_n^{-1}\| \leq \frac{1}{1-\|I-a_n\|} \leq \frac{1}{1-\frac{1}{2}} = 2$, so $\|a_n^{-1}\|$ is bounded. □

Now let A be a Banach algebra with identity I, and let a be an element of A. We defined the spectrum of a to be the set of all scalars λ such that $a - \lambda I$ is not invertible in A. The complement in $\mathbb{C}$ of the spectrum of a is called the *resolvent set* of a, and the function $r_a : \lambda \to (a - \lambda I)^{-1}$ is called the *resolvent function* of a. The function $\lambda \to a - \lambda I$ is clearly continuous from $\mathbb{C}$ to A. Since inversion is continuous on the set of invertible elements, the resolvent function is continuous on the resolvent set.

Proposition 22 *Let a be an element of a Banach algebra A with identity I. If $|\lambda| > \|a\|$, then λ lies in the resolvent set of a. In particular, the spectrum of a is a subset of the disc in $\mathbb{C}$ of radius $\|a\|$ centered at the origin.*

Proof: If $|\lambda| > \|a\|$, then $\lambda \neq 0$ and $\|\frac{a}{\lambda}\| < 1$, so $I - \frac{a}{\lambda}$ is invertible, with inverse $\sum_{k=0}^{+\infty}(\frac{a}{\lambda})^k$. Thus $\lambda I - a = \lambda(I - \frac{a}{\lambda})$ is invertible with inverse $\sum_{k=0}^{+\infty} \frac{a^k}{\lambda^{k+1}}$. It follows that $a - \lambda I$ is invertible. □

Corollary 16 *Let a be an element of a Banach algebra A with identity I. Then the spectrum of a is compact.*

Proof: It suffices to show that the spectrum is closed, i.e., that the resolvent set is open. Suppose λ is an element of the resolvent set. Now $(a - \mu I) - (a - \lambda I) = (\lambda - \mu)I$. Thus if $|\lambda - \mu|$ is sufficiently small, then $a - \mu I$ is also invertible, since the invertible elements of A form an open set. Thus each element of the resolvent set is an interior point of the resolvent

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set. □

The next proposition tells us that the resolvent function is *weakly analytic*.

Proposition 23 *Let ϕ be a bounded linear functional on A. Then the function $\phi \circ r_a$ is differentiable everywhere on the resolvent set of a.*

Proof: Suppose μ is an element of the resolvent set. Then the difference quotient for $\phi \circ r_a$ at μ is

$$\begin{aligned}\frac{\phi \circ r_a(\lambda) - \phi \circ r_a(\mu)}{\lambda - \mu} &= \phi\left(\frac{(a-\lambda I)^{-1} - (a-\mu I)^{-1}}{\lambda - \mu}\right) \\ &= \phi\left\{(a-\lambda I)^{-1}\frac{(a-\mu I)-(a-\lambda I)}{\lambda-\mu}(a-\mu I)^{-1}\right\} \\ &= \phi\{(a-\lambda I)^{-1}(a-\mu I)^{-1}\}.\end{aligned}$$

As $\lambda \to \mu$, this last tends to $\phi\{(a-\mu I)^{-2}\}$, so $\phi \circ r_a$ is differentiable at μ. □

Remark 7 In fact the argument of the last proposition shows that r_a is a Banach algebra-valued function which is analytic on the resolvent set of a, with derivative $(a - \mu I)^{-2}$.

Theorem 12 *Let A be a Banach algebra with identity I. For each a in A, the spectrum of a is a non-empty compact subset of the disc in $\mathbb{C}$ with radius $\|a\|$ centered at the origin.*

Proof: It remains only to show that the spectrum is non-empty. Suppose then that the spectrum of some a in A is void. Then for each bounded linear functional ϕ on A, $\phi \circ r_a$ is analytic on the entire complex plane. Now for non-zero λ,

$$\phi \circ r_a(\lambda) = \phi((a-\lambda I)^{-1}) = \frac{1}{\lambda}\phi\left(\left(\frac{a}{\lambda} - I\right)^{-1}\right).$$

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As the modulus of λ tends to infinity, $\phi\left(\left(\frac{a}{\lambda} - I\right)^{-1}\right)$ tends to $\phi(I)$, by continuity of inversion and continuity of ϕ. It follows that $\phi \circ r_a$ tends to zero as the modulus of λ tends to infinity. Thus we may choose a compact disc D such that $\phi \circ r_a$ is arbitrarily small off D. Since $\phi \circ r_a$ is continuous, it is bounded on D, and hence on the whole of the plane. Thus $\phi \circ r_a$ is a bounded entire function, so by Liouville's Theorem it is constant. But $\phi \circ r_a$ tends to zero at infinity, so it must be the constant function zero, and this must hold for all bounded linear functionals on A.

Thus we have established that for all ϕ in the dual space of A and for all complex λ,

$$\phi \circ r_a(\lambda) = \phi((a - \lambda I)^{-1}) = 0.$$

Now it follows from the Hahn-Banach Theorem that if x is any non-zero element of A, then there exists a bounded linear functional ϕ on A such that $\phi(x)$ is not zero. This means that the resolvent function of a must be zero for every complex λ, which is clearly absurd since no invertible element can ever be zero. This contradiction shows that the spectrum of a is non-empty after all, and the proof is complete. □

Corollary 17 *Let A be a normed algebra with identity I over the complex numbers. Then for each a in A the spectrum of a is not empty.*

Proof: Let a be an element of A, and let B be a completion for A. (See the exercise below.) We identify A with its (isomorphic) image in B. Then since A is a subalgebra of B and the identity of B lies in A, we have $\mathrm{sp}_A(a) \supseteq \mathrm{sp}_B(a)$. Since B is a Banach algebra, $\mathrm{sp}_B(a)$ is not empty, so it follows immediately that $\mathrm{sp}_A(a)$ is not empty. □

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Exercise Let A be a normed algebra over the complex numbers. Show that there exist a Banach algebra B and an isometric algebra homomorphism L from A into B such that $L(A)$ is a dense subalgebra of B and L is an algebra isomorphism of A onto its image. (Hint: use the canonical embedding of A into its Banach space double dual, and take B to be the closure of the image of A under the embedding.) Show further that if A has an identity I, then $L(I)$ is an identity for B. Show finally that if C and M are any pair with the same properties as B and L, then there exists a unique isometric algebra isomorphism U from B onto C such that $M = U \circ L$. (The algebra B is called a *completion* of A.)

Example 3 The spectrum of a square matrix is its set of eigenvalues. If S is the left shift on $\ell^2(N)$, then for every λ with $|\lambda| < 1$, the geometric sequence $\{1, \lambda, \lambda^2, \dots\}$ is an eigenvector for S with eigenvalue λ, so the spectrum of S contains the open unit disc; since the spectrum of an operator is always closed and is bounded by the norm of the operator, the spectrum of the left shift is exactly the closed unit disc. Since the right shift is the adjoint of the left shift, the spectrum of the right shift is the set of all complex conjugates of elements of the closed unit disc; this means of course that the spectrum of the right shift is also the closed unit disc. If X is a compact Hausdorff space, then the spectrum of any f in $C(X)$ is its range, since $f - \lambda 1$ is invertible exactly when f never assumes the value λ. We shall see below that the spectrum of any function in $L^\infty[0, 1]$ is its essential range.

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Exercises

1. Verify the assertions made in this section about examples of algebras, ideals, normed algebras, and Banach algebras. In particular show that the ring of quaternions is an algebra over $\mathbb{R}$ but not an algebra over $\mathbb{C}$.

2. Let A be a normed algebra with identity I and norm $\| \ \|$, and define a function N on A by $N(a) = \sup\{\|ab\| : \|b\| \leq 1\}$. Show that N is a norm on A, that N and $\| \ \|$ are equivalent, and that $N(I) = 1$.

3. Let A be a normed algebra (resp., Banach algebra), and define A_1 to be $A \times \mathbb{C}$, with norm given by $\|(x, \alpha)\| = \|x\| + |\alpha|$ and with the usual product vector space structure. Define a multiplication in A_1 by $(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$.

   a) Show that A_1 is a normed algebra (resp., Banach algebra) with identity $(0, 1)$, and that A is isometrically isomorphic as an algebra to the ideal $A \times \{0\}$ of A_1.

   b) What happens if the algebra A already has an identity?

4. Let A be a Banach algebra with identity, and let a be an element of A. Use the Fundamental Theorem of Algebra to show that for each polynomial p with complex coefficients we have $sp(p(a)) = p(sp(a))$, where $p(sp(a)) = \{p(\lambda) : \lambda \in sp(a)\}$.

5. Let a and b be elements of a Banach algebra A with identity. Show that the non-zero elements of $sp(ab)$ and $sp(ba)$ coincide. Give an example with $0 \in sp(ab)$ but $0 \notin sp(ba)$.

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6. Let A be a normed algebra with identity I.

   a) Show that there exist a Banach algebra B and an isometric linear function L from A into B such that $L(A)$ is a dense subalgebra of B and L is an algebra isomorphism of A onto its image $L(A)$. Show further that $L(I)$ is an identity for B. [Hint: Embed A into its Banach space double dual and pass to the closure of the image of this embedding.]

   b) Show that if C and M are any pair with the same properties as B and L, then there exists a unique isometric algebra isomorphism U from B to C such that U following L is M.

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Division algebras and the Gelfand-Mazur Theorem

A *division ring* is a ring with identity in which every non-zero element has a multiplicative inverse. Every field is a division ring, and a division ring that is commutative is a field. A *division algebra* is an algebra that is also a division ring.

The quaternions form a non-commutative division algebra over the real numbers (but not over the complex numbers, since they fail to be an algebra over $\mathbb{C}$). It is actually moderately difficult to find examples of non-commutative division rings, since there are several general results to the effect that a division ring or division algebra with some additional property is actually a field. It is, for example, a theorem of Wedderburn that a finite division ring is a field [JNI, p. 431]. Another theorem to this effect is even more restrictive: the Gelfand-Mazur Theorem [MS, Gelfand] asserts that a normed division algebra over $\mathbb{C}$ is naturally isomorphic to $\mathbb{C}$.

To prove the Gelfand-Mazur Theorem, we begin with a few observations about spectra of elements in division algebras. Suppose a is an element in a division algebra A, and suppose that λ is an element of the spectrum of a. Then $a - \lambda I$ is not invertible, so must be zero, so $a = \lambda I$. It follows that for each element a of a division algebra, either the spectrum of a is empty, or the spectrum of a is a singleton set $\{\lambda\}$, in which case $a = \lambda I$.

Now let A be a normed division algebra over $\mathbb{C}$, so that the spectrum of each element of A is non-empty. Then each element is a scalar multiple of the identity in a unique way, and it is easy to check that the function

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$\lambda \to \lambda I$ is an algebra isomorphism of $\mathbb{C}$ onto A. (It is in fact the unique such isomorphism.)

Theorem 13 (Gelfand-Mazur) *If A is a normed division algebra over the complex numbers, then A is naturally isomorphic (as an algebra) and isometric to the complex numbers.*

Proof: The remarks above establish the existence of a unique algebra isomorphism of $\mathbb{C}$ onto A, given by $\lambda \to \lambda I$. Since the norm of an identity is always 1, this map is isometric. □

Now for any Banach algebra A and any closed ideal J in A, the quotient space A/J is an algebra under the usual operations on cosets and a Banach space under the norm

$$\|a + J\| = \inf\{\|a + j\| : j \in J\}.$$

It is easy to check that this norm makes A/J into a Banach algebra. The quotient mapping $Q(a) = a + J$ is thus a norm-decreasing algebra homomorphism of A onto A/J, since $\|a + J\| \leq \|a + 0\| = \|a\|$.

Proposition 24 *If J is an ideal in a Banach algebra A, then the closure of J is also an ideal in A. In particular, if J is a maximal ideal in A, then J is either closed or dense in A. If A has an identity, then any maximal ideal in A is closed.*

Proof: If j is an element of the closure of J, choose $\{j_n\}$ in J with j_n convergent to j. Then for all a in A, aj_n and j_na converge respectively to aj and ja. It follows that the closure of J is an ideal in A. If J is maximal,

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then it is maximal with respect to being a proper ideal of A, so its closure must agree either with J or with A. Suppose now that I is an identity of A. Then I is an interior point of the set of all invertible elements of A. Thus if an ideal J is dense, then J must meet this set of invertible elements; but an ideal that contains an invertible element contains the identity, and hence is all of A. It follows that no proper ideal of A can be dense. In particular, no maximal ideal of a Banach algebra with an identity can be dense. □

Recall from algebra [JNI, Sections 2.5–2.7] that a commutative ring R with identity is a field if and only if it has no ideals other than $\{0\}$ and R, and that an ideal in a commutative R is maximal if and only if the quotient ring is a field. It follows that if M is a maximal ideal in a commutative Banach algebra with identity, then A/M is both a field and a Banach algebra; by the Gelfand-Mazur Theorem, A/M is isometrically isomorphic to $\mathbb{C}$ under a unique isomorphism κ. Since κ is isometric, the composition $\kappa \circ Q$ of the quotient map Q and κ is a norm-decreasing algebra homomorphism of A onto $\mathbb{C}$ with kernel equal to M.

Definition 15 A *character* of a Banach algebra A is a non-zero algebra homomorphism of A into $\mathbb{C}$.

The Gelfand-Mazur Theorem thus implies that every maximal ideal in a commutative Banach algebra A with identity is the kernel of a character. We shall see below that this association of a maximal ideal with a character is actually a bijection of the set of all maximal ideals in A with the set of all characters on A. We end this section by establishing that for commutative algebras A, maximal ideals and characters always exist.

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Proposition 25 *Let A be a commutative Banach algebra with identity. Then every proper ideal of A is contained in a maximal ideal of A.*

Proof: Let J be a proper ideal of A. Order the set S of all proper ideals in A that contain J by set theoretic inclusion. Since the identity is in no proper ideal, the union U of any chain of ideals in S is properly contained in A. Since it is the union of a chain, U is an ideal. Thus each chain in S has an upper bound in S, so by Zorn's Lemma, S has a maximal element. It is easy to see that such a maximal element is a maximal ideal of A and contains J. □

If A is a commutative ring with identity and a is an element of A, then the ideal generated by a is $aA = \{ab : b \in A\}$. It follows easily that the ideal generated by a is proper if and only if a is not invertible. We thus have the following result.

Corollary 18 *If A is a commutative Banach algebra with identity, then each non-invertible element of A lies in a maximal ideal of A. In particular, for each non-invertible element a of A there exists a character of A that vanishes on a.*

Exercise Show that the algebra A of all two by two matrices over the complex numbers has no characters, and that $\{0\}$ is a maximal ideal in A.

Thus there exist (non-commutative) Banach algebras with an identity that have no characters. Although every Banach algebra with identity has at least one maximal ideal, the absence of characters is typical of the non-commutative ones. One can show in fact that for any Hilbert space $\mathcal{H}$, the

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compact operators on $\mathcal{H}$ form a simple algebra that has no characters unless $\mathcal{H}$ is one dimensional.

Exercises

1. Let A be a Banach algebra, and let J be a closed ideal of J. Verify that the norm

$$\|a + J\| = \inf\{\|a + j\| : j \in J\}$$

makes A/J into a Banach algebra.

2. Let A be the algebra of all two by two matrices over the complex numbers. Show that A has no characters and that $\{0\}$ is a maximal ideal in A.

3. Let m and n be strictly positive integers, and let $A = M_m(\mathbb{C}) \oplus M_n(\mathbb{C})$.

a) Show that A is naturally isomorphic (as an algebra) to a subalgebra of $M_{m+n}(\mathbb{C})$.

b) What are the ideals of A?

c) Find a non-commutative Banach algebra with identity that has exactly one character.

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Ideals and characters

We saw in the last section that every maximal ideal in a commutative Banach algebra with identity is the kernel of a character. It turns out that in any Banach algebra (commutative or not) an ideal can be the kernel of at most one character.

Proposition 26 *Let V be a vector space, and let φ be a linear functional on V. Suppose ψ is a linear functional which vanishes on the kernel of φ. Then ψ is a scalar multiple of φ.*

Proof: We may assume that φ is non-zero, so that φ maps V onto the scalar field F. Let K be the kernel of φ. By the homomorphism theorem, the map $j : x + K \to \varphi(x)$ is an isomorphism of V/K onto F. Since ψ vanishes on K, the linear map $\psi' : x + K \to \psi(x)$ of V/K into F is well defined, so the composition $\psi' \circ j^{-1}$ is a well defined linear map of F into F, and hence is multiplication by an element λ of F. Now for each x, $\psi' \circ j^{-1}$ takes $\varphi(x)$ to $x + K$ to $\psi(x)$, so $\psi(x) = \lambda\varphi(x)$ for all x in V. $\square$

Proposition 27 *Let A be an algebra with identity I over a field F, and let φ be a non-zero algebra homomorphism of A into F. Then $\varphi(I) = 1$.*

Proof: Since φ is linear it must map onto F. Thus φ is a ring homomorphism of A onto F. Now if φ is any surjective homomorphism of a ring R with identity I to a ring S, then $\varphi(I)$ is an identity for S, since $\varphi(a)\varphi(I) = \varphi(aI) = \varphi(a) = \varphi(Ia) = \varphi(I)\varphi(a)$ for every a in R. The proposition now follows from the uniqueness of the multiplicative identity in the ring F. $\square$

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Corollary 19 *Let A be a Banach algebra with identity I, and let φ be a character of A. Then $\varphi(I) = 1$. Moreover, if K is the kernel of φ, and if ψ is a character of A with kernel K, then $\psi = \varphi$.*

Proof: By the last proposition, we have $\varphi(I) = 1$. Since ψ and φ have the same kernel, ψ is a scalar multiple of φ. But we also have $\psi(I) = 1$, so $\psi = \varphi$. □

Thus each maximal ideal in a commutative Banach algebra with identity is the kernel of a unique character. The next proposition asserts that every character arises in this way.

Proposition 28 *Let φ be a character of a Banach algebra A with identity I. Then the kernel of φ is a maximal ideal of A.*

Proof: The kernel of an algebra homomorphism is always an ideal. Since $\varphi(I) = 1$, the kernel of φ is a proper ideal. If now ψ is a homomorphism of R onto S with kernel K, then the map $J \to \psi^{-1}(J)$ is an order preserving map of the set of all ideals of S onto the set of all ideals of R which contain K. Since $\mathbb{C}$ has no non-trivial ideals, there are no ideals between the kernel of φ and A. □

Corollary 20 *Let A be a commutative Banach algebra with identity. The function $\varphi \to$ kernel(φ) is a bijection from the set $\chi(A)$ of all characters on A to the set Max(A) of all maximal ideals of A.*

Proof: If φ is a character on A, its kernel is a maximal ideal. Since A is commutative, each maximal ideal is the kernel of a character on A. Thus the

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map $\phi \to \text{kernel}(\varphi)$ takes $\chi(A)$ onto $\text{Max}(A)$. Since a maximal ideal is the kernel of at most one character, the map is one-to-one. $\square$

Recall that if A is a Banach algebra with identity, then every maximal ideal in A is closed. Since a linear functional on a Banach space is bounded if and only if its kernel is closed (exercise), the last proposition implies that every character φ of such an algebra A is bounded. A great deal more is true, and hinges on the fact that for each a in A, each character φ of A, and each scalar λ, we have $\varphi(a - \lambda I) = 0$ if and only if $\varphi(a) = \lambda$.

Proposition 29 *Let A be a Banach algebra with identity I, and let φ be a character on A. Then for each a in A, $\varphi(a)$ is an element of the spectrum of a. Moreover, φ is bounded, and $\|\varphi\| = 1$.*

Proof: For each a in A we have $\varphi(a - \varphi(a)I) = \varphi(a) - \varphi(a)\varphi(I) = \varphi(a) - \varphi(a)1 = 0$. Since φ maps I to 1, it maps no invertible element to zero. Thus $\varphi(a)$ lies in the spectrum of a. In particular, $|\varphi(a)| \leq \|a\|$. Thus $\|\varphi\| \leq 1$. Since $\varphi(I) = 1$, $\|\varphi\| = 1$. $\square$

Corollary 21 *Let A be a commutative Banach algebra with identity I. Then for each a in A, $sp(a) = \{\varphi(a) : \varphi \in \chi(A)\} = \{\varphi(a) : \varphi$ is a character of $A\}$.*

Proof: Let a be an element of A. By the last proposition, $\text{sp}(a) \supseteq \{\varphi(a) : \varphi \in \chi(A)\}$. Recall from the last chapter that if A is commutative, then for each non-invertible element x of A there exists a character φ of A such that $\varphi(x) = 0$. In particular, if $\lambda \in \text{sp}(a)$, then there exists a character φ of A such that $\varphi(a - \lambda I) = 0$, i.e., such that $\varphi(a) = \lambda$. Thus $\text{sp}(a) \subseteq \{\varphi(a) : \varphi \in \chi(A)\}$. $\square$

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Corollary 22 *Let A be a commutative Banach algebra with identity I. Then for each a in A and each polynomial p, sp$(p(a)) = \{p(\lambda) : \lambda \in \mathrm{sp}(a)\}$. In particular, for each $n \geq 1$,* $\mathrm{sp}(a^n) = \{\lambda^n : \lambda \in \mathrm{sp}(a)\}$.

Proof: It is easy to verify that for each a in A, each polynomial p, and each character φ of A we have $\varphi(p(a)) = p(\varphi(a))$. (Here $p(a)$ has the obvious meaning: if p is a polynomial in the variable t, then substitute the element a for t and carry out in the algebra A the operations indicated by any formula for p.) Thus $\mathrm{sp}(p(a)) = \{\varphi(p(a)) : \varphi \in \chi(A)\} = \{p(\varphi(a)) : \varphi \in \chi(A)\} = \{p(\lambda) : \lambda \in \mathrm{sp}(A)\}$. $\square$

This last corollary is the first of several "spectral mapping theorems," each of which asserts that for some class of functions f and for all elements in a certain kind of algebra, or for all elements with certain properties, we have $\mathrm{sp}(f(a)) = \{f(\lambda) : \lambda \in \mathrm{sp}(a)\}$. In particular, we shall see that if A is a commutative C^*-algebra with identity (see below), then for all a in A and all functions f that are continuous on the spectrum of a, $\mathrm{sp}(f(a)) = \{f(\lambda) : \lambda \in \mathrm{sp}(a)\}$. This implies that for any normal operator T on a Hilbert space and any continuous function f on the spectrum of T we have $\mathrm{sp}(f(T)) = \{f(\lambda) : \lambda \in \mathrm{sp}(T)\}$. For normal operators we shall also obtain a spectral mapping theorem for measurable functions. For the present, we shall be content with removing the hypothesis of commutativity in the last corollary above.

Proposition 30 *Let A be a Banach algebra with identity I. Then for each a in A and each polynomial p,* $\mathrm{sp}(p(a)) = \{p(\lambda) : \lambda \in \mathrm{sp}(a)\}$.

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Proof: If p has degree zero, then $p = c$ for some scalar c, and $p(a) = cI$. In this case the spectrum of $p(a)$ is $\{c\}$, and $p(\lambda) = c$ for every λ in the spectrum of a. Thus we may assume that p has degree at least one. Let μ be a scalar, and let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be the roots (not necessarily distinct) of $p(\lambda) - \mu$, so that $p(a) - \mu I = c(a - \lambda_1 I)(a - \lambda_2 I) \cdots (a - \lambda_n I)$ with $c \neq 0$. Observe that each factor in this product commutes with every other factor. If now $p(a) - \mu I$ is invertible, then $I = x[c(a - \lambda_1 I) \cdots (a - \lambda_n I)]$ implies that $(a - \lambda_n I)$ has a left inverse, and $I = [c(a - \lambda_1 I) \cdots (a - \lambda_n I)]x$ implies that $(a - \lambda_1 I)$ has a right inverse. Since all the factors commute, this means that each factor has both a left inverse and a right inverse, so that each factor is invertible. Thus if $p(a) - \mu I$ is invertible, then each factor $(a - \lambda_i I)$ is invertible. Conversely, if each $(a - \lambda_i I)$ is invertible, then $p(a) - \mu I$ is a product of invertible elements, and so is itself invertible. If follows that μ is not in the spectrum of $p(a)$ if and only if each λ_i is not in the spectrum of a. Thus μ lies in the spectrum of $p(a)$ if and only if one of the λ_i lies in the spectrum of a. Since the λ_i are all the roots of $p(\lambda) - \mu$, this says that μ lies in the spectrum of $p(a)$ if and only $\mu = p(\lambda)$ for some λ in the spectrum of a. □

Exercises

1. Let f be a linear functional on a Banach space. Show that f is continuous if and only if its kernel is closed.

2. Let $\mathcal{H}$ be infinite dimensional, and let A be the set of all operators in $\mathcal{B}(\mathcal{H})$ of the form $T + \lambda I$, where T is compact and λ is a scalar.

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a) Show that A is a $*$-stable subalgebra of $\mathcal{B}(\mathcal{H})$ and that the representation of an element of A in the form $T + \lambda I$ is unique.

b) Show that $f : T + \lambda I \to \lambda$ is a continuous linear functional on A. (Use the previous exercise.)

c) Show that A is norm-closed in $\mathcal{B}(\mathcal{H})$.

d) Show that $f : T + \lambda I \to \lambda$ is a character of the algebra A.

Note: When we have developed a measurable functional calculus (see "Applications of the Measurable Functional Calculus" below), we shall see that this function is the unique character on the algebra A.

3. Let D be the closed unit disc in the complex plane, and let A be the subalgebra of $C(D)$ consisting of all those functions that are analytic in the interior of D. Show that if w is a point of D, then the function $\varphi_w : f \to f(w)$ is a character on A. Show further that if w is an interior point of D, then the kernel of φ_w is the principal ideal $\{(z - w)h(z) : h \in A\}$ of A generated by the function $z - w$.

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An Example: $\ell^1(\mathbb{Z})$ and the Wiener Algebra

We pause briefly in our development of a general theory of commutative Banach algebras to take up an important example. We shall show in this section that $\ell^1 = \ell^1(\mathbb{Z})$ is a commutative Banach algebra with identity in a natural way. In fact, ℓ^1 contains a copy of the integers, and we shall use the group operation in the integers to define the multiplication on ℓ^1. For later purposes it will be convenient for us to include here some results about the ℓ^p spaces in general. We shall need the fact (very easily checked) that for any Banach space X, the space $\mathcal{B}(X)$ is a Banach algebra under the operator norm $\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|$.

Recall that the functions e_n, defined by

$$e_n(n) = 1 \quad \text{and} \quad e_n(m) = 0 \qquad \text{for } m \neq n,$$

lie in ℓ^p for every p, where $1 \leq p \leq +\infty$. It is not difficult to show that for $1 \leq p < +\infty$, each $x = \{x(n)\}$ in ℓ^p has a unique representation

$$x = \sum_{n=-\infty}^{+\infty} x(n) e_n$$

as an element of the closed linear span of the e_n.

Definition 16 For each integer m and each p with $1 \leq p \leq +\infty$, we define a *translation operator* T_m on ℓ^p by

$$(T_m x)(n) = x(n-m).$$

It is easy to check that for each m and n we have $T_n T_m = T_{n+m} = T_m T_n$, and that $T_m e_n = e_{n+m}$. In particular, we have $T_n T_m = T_{n+m}$ for all m and n.

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It is also easy to check that each T_m is a linear isometry of ℓ^p onto itself. It follows readily that for each function f in ℓ^1 the series $\sum_m f(m)T_m$ converges absolutely in $\mathcal{B}(\ell^p)$ to an operator T_f, and that $\|T_f\| \leq \sum_m |f(m)| = \|f\|_1$. Note that $T_{e_m} = T_m$ for all integers m and that T_0 is the identity operator I on ℓ^p. (Here the subscripted zero denotes the integer zero and not the ℓ^1-function which is everywhere zero on $\mathbb{Z}$.)

Thus we have a norm-decreasing linear map $T : f \to T_f$ from ℓ^1 into the Banach algebra $\mathcal{B}(\ell^p)$. For any particular f in ℓ^1 and g in ℓ^p, we have for all integers k that

$$(T_f g)(k) = \sum_m f(m)T_m g(k) = \sum_m f(m)g(k-m).$$

Since $T_f(e_0) = \sum_m f(m)T_m(e_0) = \sum_m f(m)e_m = f$, the map T is injective for every p and isometric when $p = 1$. We thus have the following result.

Proposition 31 *The map T of the preceding paragraphs is an injective norm-decreasing linear map of ℓ^1 into $\mathcal{B}(\ell^p)$, and T is isometric if $p = 1$. Moreover, for all f in ℓ^1, we have $T_f(e_0) = f$ and $T_{e_0} = T_0 = I$.*

Now let $p = 1$. Since T_n and T_m commute for every m and n, it follows easily that the span S_0 of the T_m is a commutative subalgebra of $\mathcal{B}(\ell^1)$. Since the span of the e_n is dense in ℓ^1 and T is an isometry, S_0 is dense in $T(\ell^1)$. In a Banach algebra the closure of a commutative subalgebra is again a commutative subalgebra, so $T(\ell^1)$ is a closed subalgebra of $\mathcal{B}(\ell^1)$. It follows then that for each f and g in ℓ^1, $T_f T_g = T_h$ for some function h in ℓ^1. From the proposition above, $h = T_h(e_0)$.

Definition 17 Let f and g be elements of ℓ^1. We define the *convolution* of

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f with g, denoted by $f * g$, by

$$f * g = T_f T_g e_0 = T_f g.$$

Convolution is thus the operation on ℓ^1 that corresponds, under the linear isometry T, to the algebra multiplication in $T(\ell^1)$. In particular, T is an isometric algebra isomorphism of ℓ^1 onto $T(\ell^1)$, and ℓ^1 is thus a commutative Banach algebra with identity $e_0 = T^{-1}(I)$. Since ℓ^1 is commutative, we have established the following result.

Proposition 32 *For any functions f and g in ℓ^1, we have*

$$(f * g)(k) = \sum_{m=-\infty}^{+\infty} f(m)g(k-m) = \sum_{n=-\infty}^{+\infty} f(k-n)g(n) = (g * f)(k).$$

In particular, these series are absolutely convergent for each k, and the resulting function $f * g$ lies in ℓ^1.

Remark 8 Note that for all m and n, $e_m * e_n = e_{m+n}$, so that the multiplicative structure of ℓ^1 contains a copy of the (additive) group $\mathbb{Z}$. The equality of the two series in the proposition reflects the commutativity of convolution. The reader is invited to use one of the expressions $\sum_{m=-\infty}^{+\infty} f(m)g(k-m)$ and $\sum_{n=-\infty}^{+\infty} f(k-n)g(n)$ to define the convolution of f and g, and to prove directly that it makes ℓ^1 into a commutative Banach algebra with identity e_0. The necessary verifications all follow from absolute summability. Much of the development above can be done with a general group G in place of Z, although the resulting algebra $\ell^1(G)$ will be commutative only when the group G is commutative. See for example [RWI].

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Let us next identify the character space of ℓ^1. The key to this identification is the fact that the element e_1 generates ℓ^1 as an algebra, since $e_n = [e_1]^n$ for all n in Z.

Proposition 33 *For each complex number z of modulus one, the function φ_z, defined by $\varphi_z(f) = \sum_{n=-\infty}^{+\infty} f(n)z^n$, is a character on ℓ^1. Moreover, every character on ℓ^1 arises in this way, and the function $z \to \varphi_z$ is a bijection of the unit circle in $\mathbb{C}$ onto the character space of ℓ^1.*

Proof: It is easy to check that each φ_z is linear and multiplicative, and φ_z is non-zero since $\varphi_z(e_1) = z$.

Conversely, suppose φ is a character of ℓ^1, and let $z = \varphi(e_1)$. Since e_1 is invertible (with inverse e_{-1}), z is non-zero. Since characters are norm-decreasing, we have $|\varphi(e_1)| \leq 1$ and $|\varphi(e_{-1})| \leq 1$. But $\varphi(e_{-1}) = \frac{1}{z}$, since $e_{-1} = (e_1)^{-1}$ and φ is multiplicative. Thus z and $\frac{1}{z}$ both have modulus less than or equal to one, and it follows that $|z| = 1$. For all n we have $\varphi(e_n) = [\varphi(e_1)]^n = z^n$, since φ is multiplicative. It then follows from the linearity and continuity of φ that $\varphi(f) = \sum_{n=-\infty}^{+\infty} f(n)z^n$ for all f in ℓ^1, so φ must coincide with φ_z.

In particular, every character on ℓ^1 arises as φ_z for some z in the unit circle, so the map $z \to \varphi_z$ is onto. Since e_1 generates ℓ^1, two characters φ_z and φ_w agree on the whole of ℓ^1 if and only if they agree on e_1. It follows then that characters φ_z and φ_w agree on ℓ^1 if and only if $z = w$, and the proof is complete. $\square$

The algebra ℓ^1 also has another important realization. Indeed, suppose that $\{a_n\}$ is a sequence in ℓ^1, and let us form the series $\sum_{n=-\infty}^{+\infty} a_n z^n$. If we

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confine the values of z to the unit circle T, then the partial sums of this series are uniformly and absolutely Cauchy, and hence converge uniformly and absolutely to a function f, which must be continuous on T by the uniformity of the convergence. It is easy to check that the Fourier coefficients of f are exactly the sequence $\{a_n\}$. Since any other function on T with these same Fourier coefficients is the limit in $L^2(T)$ of these same partial sums, we see that each element of ℓ^1 is the sequence of Fourier coefficients for a unique continuous function on T, a function to which the Fourier series actually converges uniformly. Conversely, if f is a continuous function on T with absolutely summable Fourier coefficients $\{a_n\}$, then the series $\sum_{n=-\infty}^{+\infty} a_n z^n$ converges uniformly on T to f, by the same argument. Thus we have a one-to-one correspondence, given by

$$a = \{a_n\} \to \hat{a}, \quad \text{where } \hat{a}(z) = \varphi_z(a) = \sum_{n=-\infty}^{+\infty} a_n z^n,$$

between ℓ^1 and the subset of $C(T)$ consisting of those functions with absolutely summable Fourier coefficients. This correspondence is the *Fourier transform* on ℓ^1. Since each character φ_z on ℓ^1 is multiplicative, we have

$$\{a_n\} * \{b_n\} \to \hat{a}\hat{b},$$

so that the Fourier transform is an injective algebra homomorphism into $C(T)$. Thus we may identify ℓ^1, via the Fourier transform, with the *Wiener algebra*, i.e., the subalgebra of $C(T)$ consisting of all those functions with absolutely summable Fourier coefficients. In the next section, we will obtain a remarkable theorem of Wiener concerning invertibility in the Wiener algebra.

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Exercises

1. Show that for any Banach space X, $\mathcal{B}(X)$ is a Banach algebra with identity under the operator norm $\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$.

2. Verify that for $1 \leq p < +\infty$, each $x = \{x_n\}$ in ℓ^p has a unique representation in the form $x = \sum_{-\infty}^{+\infty} \alpha_n e_n$ with $\alpha_n \in \mathbb{C}$ for all n.

3. Verify that for $1 \leq p \leq +\infty$ and for $m \in \mathbb{Z}$, the equation $(T_m x)(n) = x(n-m)$ defines a linear isometry T_m of ℓ^p onto ℓ^p.

4. Prove directly from the definition $(f * g)(k) = \sum_{m=-\infty}^{+\infty} f(m)g(k-m)$ that $\ell^1(\mathbb{Z})$ is a commutative Banach algebra with identity e_0. (This exericise is tedious. The approach adopted in the text above is designed to give a more illuminating definition for convolution, *and* to avoid the tedium.)

5. Verify that for each z with $|z| = 1$ the function $\sum_{-\infty}^{+\infty} f(n)e_n \to \sum_{-\infty}^{+\infty} f(n)z^n$ is a character on $\ell^1(\mathbb{Z})$.

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The Gelfand transform

Among commutative Banach algebras with an identity, the algebra $C(X)$ plays a central role. Indeed, $C(X)$ is the natural representing object for the class. To see why this is so, we consider an analogous representation from the theory of Banach spaces.

If B is a Banach space and K is the closed unit ball in its dual space B^*, then by Alaoglu's Theorem K is a weak* compact Hausdorff space. Each element b of B acts naturally on K (or on the whole of B^*) as a continuous function $\hat{b} : \varphi \to \varphi(b) = \hat{b}(\varphi)$. Thus we have a mapping $\Phi : b \to \hat{b} = \Phi(b)$ from B into $C(K)$. This map is clearly linear, and it follows from the Hahn-Banach Theorem that for each b in B there exists an element φ of K such that $\varphi(b) = \|b\|$. Since the norm on $C(K)$ is the supremum norm, this means that φ is an isometric map. Thus every Banach space is isometrically isomorphic to a closed subspace of a Banach space of the form $C(K)$, where K is a compact Hausdorff space. We can therefore study a general Banach space by using this representation to compare it to a $C(K)$-space. (Some information may be difficult to recover when we pass from B to its image in $C(K)$, but this approach has nevertheless been rather successful.)

We ask now whether a similar embedding is possible for a Banach algebra A. We would want the embedding to be an algebra homomorphism. Since $C(K)$ is commutative, it is clear that we cannot hope that the embedding will preserve much of the algebraic structure unless A is commutative, so we restrict our attention to commutative Banach algebras with identity I, and we ask the following questions.

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1) When is there a nice algebra homomorphism of A into an algebra of the form $C(X)$, and what should we use for X?

2) When is this embedding isometric?

3) When is this embedding an isometric isomorphism onto $C(X)$?

Gelfand [Gelfand] showed that there always exists such a homomorphism on A, and a theorem of Gelfand and Naimark (see below) answers the second and third questions as well.

We shall denote Gelfand's algebra homomorphism by Γ. Based on the example of the Banach space embedding, it seems reasonable to look for X in the Banach space dual A^* of A, and to make each $\Gamma(a)$ act on X by means of $[\Gamma(a)](\varphi) = \hat{a}(\varphi) = \varphi(a)$. Since Γ is to be an algebra homomophism, we need to have $\Gamma(ab) = \Gamma(a)\Gamma(b)$. This means that for each φ in X we need $[\Gamma(ab)](\varphi) = [\Gamma(a)\Gamma(b)](\varphi) = [\Gamma(a)](\varphi)[\Gamma(b)](\varphi)$, i.e. $\varphi(ab) = \varphi(a)\varphi(b)$. Similarly we will need to have $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\lambda\varphi(a) = \varphi(\lambda a)$, since Γ must be linear. But these conditions just say that φ is an algebra homomorphism of A into $\mathbb{C}$. Since any such homomorphism is either the zero map or a character of A, it seems reasonable to choose the character space of A for our set X.

Proposition 34 *The space of all characters of a commutative Banach algebra A with identity is a weak* compact Hausdorff topological space.*

Proof: The weak* topology on A^* is a Hausdorff topology, so the character space $\chi(A)$ is weak* Hausdorff. Since every character is an element of the norm-closed unit ball K of A^*, it suffices (by Alaoglu's Theorem) to show

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that the character space is weak* closed in K. Suppose then that $\varphi_\gamma \to \varphi$ weak* in K, and that each φ_γ is a character. Then for each a in A, $\varphi_\gamma(a) \to \varphi(a)$. In particular, for each a and b in A we have $\varphi_\gamma(ab) \to \varphi(ab)$. But $\varphi_\gamma(ab) = \varphi_\gamma(a)\varphi_\gamma(b)$ since each φ_γ is a character, and $\varphi_\gamma(a)\varphi_\gamma(b) \to \varphi(a)\varphi(b)$ by continuity of multiplication in $\mathbb{C}$. Thus $\varphi(ab) = \varphi(a)\varphi(b)$. Similarly φ is linear, and $\varphi(I) = \lim \varphi_\gamma(I) = 1 \neq 0$, so φ is a character. Thus $\chi(A)$ is weak* closed in K as desired. □

Definition 18 Let A be a commutative Banach algebra with identity. Let a be an element of A. Then the *Gelfand transform* of a is the function $\Gamma(a)$ on $\chi(A)$ to $\mathbb{C}$ defined by $[\Gamma(a)](\varphi) = \varphi(a)$. The *Gelfand transform* of the algebra A is the function Γ which maps a to $\Gamma(a)$.

The Gelfand transform $\Gamma(a)$ of a is often denoted by $\hat{a}$. Note that by definition of the weak* topology, each $\Gamma(a)$ is weak* continuous on $\chi(A)$. Thus the Gelfand transform Γ maps A into $C(\chi(A))$. By arguments indicated above, it follows directly from the definition of Γ that Γ is an algebra homomorphism of A into $C(\chi(A))$. Since the norm of any character is one, $\|\Gamma(a)\|_\infty = \sup\{|[\Gamma(a)](\varphi)| : \varphi \in \chi(A)\} = \sup\{|\varphi(a)| : \varphi \in \chi(A)\} \leq \|a\|$ for all a in A. We collect these facts in the following proposition.

Proposition 35 *Let A be a commutative Banach algebra with identity I, and let Γ be its Gelfand transform. Then Γ is a norm-decreasing algebra homomorphism from A into $C(\chi(A))$, and $\Gamma(I)$ is the constant function 1. Moreover, the kernel of Γ is the intersection of all the maximal ideals of A.*

Proof: $[\Gamma(I)](\varphi) = \varphi(I) = 1$ for all φ in $\chi(A)$, so $\Gamma(I)$ is the constant

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function 1. An element a of A lies in the kernel of Γ if and only if $\varphi(a) = [\Gamma(a)](\varphi) = 0$ for all characters φ. Since the kernels of the characters are exactly the maximal ideals of A, the kernel of Γ is the intersection of the maximal ideals. The rest of the proposition has been proved above. □

The intersection of all the maximal ideals of an algebra A with identity is called the *Jacobson radical* of A. An algebra is said to be *semisimple* if its Jacobson radical is $\{0\}$. Thus Γ is injective if and only if A is semisimple. In the next few sections we shall determine conditions that ensure that the range of Γ is dense in $C(\chi(A))$ and a further condition that ensures that Γ maps A isometrically onto $C(\chi(A))$.

One of the first successes of Gelfand's theory of commutative Banach algebras is the following characterization of invertibility, which is very easy to prove but has very far reaching implications.

Theorem 14 *Let A be a commutative Banach algebra with identity I, and let a be an element of A. Then the following statements are equivalent:*

1) a is invertible in A

2) $\Gamma(a)$ is an invertible function in $C(\chi(A))$

3) For all $\varphi \in \chi(A)$, $\varphi(a) \neq 0$.

Moreover, the range of $\Gamma(a)$ is $\mathrm{sp}(a)$.

Proof: The element a is invertible in A if and only if a lies in no maximal ideal of A if and only if $\varphi(a) \neq 0$ all characters φ of A if and only if $\Gamma(a)$ is everywhere non-zero on the character space of A. But an element of $C(\chi(A))$ is invertible in $C(\chi(A))$ if and only if it is everywhere non-zero, so 1), 2), and

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3) are equivalent. It follows that $a - \lambda I$ is not invertible if and only if there exists a character φ with $\varphi(a - \lambda I) = 0$, so $\lambda \in \mathrm{sp}(a)$ if and only if there exists a character φ with $\varphi(a) = \lambda$ if and only if there exists a character φ with $\Gamma[(a)](\varphi) = \lambda$. Thus $\lambda \in \mathrm{sp}(a)$ if and only if $\lambda \in \mathrm{range}(\Gamma(a))$. □

Example revisited: In the previous chapter we identified the character space of the algebra $\ell^1(Z)$ with the unit circle via the map $z \to \varphi_z$, where $\varphi_z(e_1) = z$. Now $[\Gamma(e_1)](\varphi_z) = \varphi_z(e_1) = z$, so $\Gamma(e_1)$ is an explicit bijection that also establishes this identification (and inverts the map $z \to \varphi_z$). Since $\Gamma(e_1)$ is continuous from the compact character space $\chi(\ell^1(\mathbf{Z}))$ into the Hausdorff unit circle, $\Gamma(e_1)$ is a closed map, and it follows that $\Gamma(e_1)$ is a homeomorphism of $\chi(\ell^1(\mathbf{Z}))$ onto the unit circle T. From the definition of the Gelfand Transform we see that for each f in $\ell^1(\mathbf{Z}))$, $[\Gamma(f)](\varphi_z) = \sum_{n=-\infty}^{+\infty} f(n)z^n$. The next result follows from the characterization of invertibility proven just above, applied to the algebra $\ell^1(\mathbf{Z})$.

Theorem 15 (Wiener's Lemma) *Let f be a continuous function on the unit circle T, and suppose that f has absolutely summable Fourier coefficients. Suppose that $f(z) \neq 0$ for all z in T. Then there exists an absolutely summable sequence $\{b_n\}$ with $\frac{1}{f(z)} = \sum_n b_n z^n$ for all z in T.*

Proof: Exercise. □

Exercise

1. Prove Weiner's Lemma.

2. Let $S = \{0, 1, 2 \ldots\}$ be the semigroup (under addition) of all non-negative integers, and let $A = \ell^1(S) = \{f : S \to \mathbb{C} : \sum_{n=0}^{+\infty} |f(n)| <$

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$+\infty\}$. Let Γ be the Gelfand transform of A.

a) Show that A is a closed subalgebra of $\ell^1(\mathbb{Z})$ and that A contains the identity of $\ell^1(\mathbb{Z})$.

b) Show that for each complex number z with $|z| \leq 1$, the function $\varphi_z : f \to \sum_{n=0}^{+\infty} f(n)z^n$ is a character on $\ell^1(S)$. Why precisely do some of those characters on $\ell^1(S)$ fail to be characters on $\ell^1(\mathbb{Z})$?

c) Show that the function $z \to \varphi_z$ is a bijection of the disc $\Delta = \{z \in \mathbb{C} : |z| \leq 1\}$ onto the character space $\chi(\ell^1(S)$ of $\ell^1(S)$ (so that every character of $\ell^1(S)$ arises as described in part b)).

d) Show that $\Gamma(e_1)$ is a homeomorphism of the character space $\chi(\ell^1(S))$ onto the disc Δ and that for all $f \in \ell^1(S)$ we have $(\Gamma(f))(\varphi_z) = \sum_{n=0}^{+\infty} f(n)z^n$. Conclude that for each f in $\ell^1(S)$, $\Gamma(f)$ is continuous on Δ and analytic in the interior $\{z : |z| < 1\}$ of Δ.

Exercise 2 was generated by a question from Allen Hoffmeyer, for which the author expresses his thanks.

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Involutions and *-algebras

We continue with our program of using the Gelfand transform to compare a general commutative Banach algebra A with identity I to the continuous functions on the character space $\chi(A)$ of A. We seek conditions under which the Gelfand transform will be an isometric isomorphism onto $C(\chi(A))$. For any compact space X, the algebra $C(X)$ has an additional structure: if $\bar{f}$ is defined by $\bar{f}(x) = \overline{f(x)}$, then the mapping $f \to \bar{f}$ resembles the adjoint map in $\mathcal{B}(\mathcal{H})$.

Definition 19 An *involution* on a complex algebra A is a function * from A to A which satisfies

a. $a^{**} = a$ for all a in A;

b. $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$ for all a and b in A and all scalars λ; and

c. $(ab)^* = b^* a^*$ for all a and b in A.

If A is a normed algebra, we shall also require that the involution be isometric:

d. $\|a^*\| = \|a\|$ for all a in A.

Examples of involutions: Conjugation on $\mathbb{C}$; the adjoint map on $\mathcal{B}(\mathcal{H})$; the map $f \to \bar{f}$ on $C(X)$; the conjugate transpose on the algebra $M_n(C)$ of all n by n complex matrices; the map $f \to f^*$, where $f^*(n) = \overline{f(-n)}$, on $\ell^1(\mathbf{Z})$.

A normed algebra with an involution is called a *normed $*$-algebra*, and if it is complete we call it a *Banach $*$-algebra*. We usually refer to a^* as

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"the adjoint of a." An element a that satisfies $a = a^*$ is said to be *self-adjoint*, and if $aa^* = a^*a$, then a is *normal*. Since $a = (a^*)^* = (Ia^*)^* = aI^*$ and $a = (a^*)^* = (a^*I)^* = I^*a$, we see that the identity of any $*$-algebra is self-adjoint.

Exercise What are the self-adjoint elements of $\ell^1(Z)$?

Definition 20 Let Φ be an algebra homomorphism between $*$-algebras A and B. If $\Phi(a^*) = \Phi(a)^*$ for all a in A, then Φ is said to be a $*$-*homomorphism*. A $*$-*isomorphism* is a bijective $*$-homomorphism.

Proposition 36 *Let a be an element of a $*$-algebra A with identity I. Then $\lambda \in \mathrm{sp}(a)$ if and only if $\bar{\lambda} \in \mathrm{sp}(a^*)$.*

Proof: Let $\lambda \in \mathbb{C}$. Then since $I^* = I$, $(a - \lambda I)b = I = b(a - \lambda I)$ if and only if $b^*(a - \lambda I)^* = I = (a - \lambda I)^*b^*$ if and only if $b^*(a^* - \bar{\lambda}I) = I = (a^* - \bar{\lambda}I)b^*$, so $a - \lambda I$ is invertible if and only if $a^* - \bar{\lambda}I$ is invertible. $\square$

In any $*$-algebra, each element has real and imaginary parts, and functionals which are real on self-adjoint elements play an important role.

Proposition 37 *Any element a in a $*$-algebra A may be written in one and only one way as $a = a_1 + ia_2$ with a_1 and a_2 self-adjoint. In fact, $a_1 = \frac{1}{2}(a + a^*)$ and $a_2 = \frac{1}{2i}(a - a^*)$.*

Proof: If $a_1 = \frac{1}{2}(a + a^*)$ and $a_2 = \frac{1}{2i}(a - a^*)$, then clearly a_1 and a_2 are self-adjoint and $a = a_1 + ia_2$. If also $a = b_1 + ib_2$ with b_1 and b_2 self-adjoint, then $a_1 - b_1 = i(b_2 - a_2)$, so that $a_1 - b_1 = (a_1 - b_1)^* = (i(b_2 - a_2))^* = -i(b_2 - a_2) = -(a_1 - b_1)$, so that $a_1 = b_1$. But then $a_2 = b_2$. $\square$

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Proposition 38 *Let A be a $*$-algebra and let ψ be a linear functional (not necessarily multiplicative or continuous) on A. Then ψ is real on each self-adjoint element of A if and only if $\psi(a^*) = \overline{\psi(a)}$ for every a in A.*

Proof: Suppose $\psi(a)$ is real whenever a is self-adjoint. Write an arbitrary a in A as $a_1 + ia_2$, with a_1 and a_2 self-adjoint. Then

$$\psi(a^*) = \psi(a_1 - ia_2) = \psi(a_1) - i\psi(a_2) = \overline{\psi(a_1) + i\psi(a_2)} = \overline{\psi(a_1 + ia_2)} = \overline{\psi(a)}.$$

If conversely $\psi(a^*) = \overline{\psi(a)}$ for every a in A, then $\psi(a) = \overline{\psi(a)}$ for every self-adjoint a in A. □

Theorem 16 *Suppose A is a commutative Banach $*$-algebra with identity I, and suppose that every element of $\chi(A)$ is real on the self-adjoint elements of A. Then Γ is a norm decreasing $*$-homomorphism from A to a dense $*$-subalgebra of $C(\chi(A))$.*

Proof: We already know that Γ is a norm decreasing algebra homomorphism. Since each φ in $\chi(A)$ is real on self-adjoint elements of A, we have $\varphi(a^*) = \overline{\varphi(a)}$ for all a in A, which means that $[\Gamma(a^*)](\varphi) = \overline{[\Gamma(a)](\varphi)} = \overline{[\Gamma(a)]}(\varphi)$ for all a in A and all φ in $\chi(A)$. Thus Γ is a $*$-homomorphism and $\Gamma(A)$ is a $*$-subalgebra of $C(\chi(A))$. But then $\Gamma(A)$ is a subalgebra of $C(\chi(A))$ which contains the constant function 1, separates the points of $(\chi(A))$, and contains $\bar{f}$ whenever it contains f. By the Stone-Weierstrass Theorem [PGK, Thm. 4.3.4, p. 146], $\Gamma(A)$ is dense in $C(\chi(A))$. □

If we could now show that Γ is an isometry, then $\Gamma(A)$ would be closed, and the last theorem above would imply that $\Gamma(A) = C(\chi(A))$. Thus our next goal is to find an additional condition that ensures that Γ preserves

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norms. We shall see that the condition we choose also ensures that each character is real on self-adjoint elements of A, i.e., that Γ preserves adjoints.

Exercises

1. What are the self-adjoint elements of $\ell^1(\mathbb{Z})$?

2. Let A be a normed $*$-algebra, and let B be the vector space direct sum of A and a copy of the complex numbers. We define a norm and product on B by

$$\|(a, \lambda)\| = \|a\| + |\lambda|$$

and

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

a. Show that with these definitions B is a normed algebra with identity $(0, 1)$.

b. Let $a \to a^*$ be an involution on A. Show that the operation $(a, \lambda) \to (a^*, \bar{\lambda})$ is an involution on B.

3. Let $J : a \to a^*$ be an involution on an algebra A. Show that J is a ring homomorphism if and only if the algebra A is commutative.

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C^*-algebras

We continue with our search for a condition that will guarantee that Γ preserves norms and adjoints. Notice that in $C(\chi(A))$, we have for every f that $\|f\bar{f}\|_\infty = \|f\|_\infty^2$. Thus if Γ preserves norms and adjoints, then $\|a^*a\| = \|\Gamma(a^*a)\|_\infty = \|\overline{\Gamma(a)}\Gamma(a)\|_\infty = \|\Gamma(a)\|_\infty^2 = \|a\|^2$ for every a in A.

Definition 21 A C^*-algebra is a Banach $*$-algebra A in which we have $\|a^*a\| = \|a\|^2$ for every element a of A.

Exercise We have assumed that the involution in a normed $*$ algebra is isometric. Show that even if we do not make this assumption, the condition "$\|a^*a\| = \|a\|^2$ for all a" implies that the involution in a C^*-algebra satisfies $\|a^*\| = \|a\|$ for all a. Show also that even if we do not assume that the norm of an identity is 1, we automatically have $\|I\| = 1$ in any C^*-algebra with identity I.

We know so far of two examples of C^*-algebras: $\mathcal{B}(\mathcal{H})$ and $C(X)$. A moment's reflection will show that any norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ (e.g., the algebra of all compact operators on $\mathcal{H}$) is also a C^*-algebra. The Gelfand-Naimark Theorem [GN] shows that up to isometric $*$-isomorphism every C^*-algebra A is such a $*$-subalgebra (for some Hilbert space $\mathcal{H}$, which depends upon A). Our immediate goal is to establish another theorem of Gelfand and Naimark (also [GN]) which asserts that for any commutative C^*-algebra A with identity, the Gelfand transformation Γ is an isometric $*$-isomorphism onto $C(\chi(A))$. This shows that up to isometric $*$-isomorphism the most general commutative C^*-algebra with identity is $C(X)$. The first

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step is to show that for C^*-algebras, Γ is always a $*$-homomorphism, i.e., that characters of commutative C^*-algebras are always real on self-adjoint elements. Since $\text{sp}(a) = \{\varphi(a) : \varphi \in \chi(A)\}$, it is sufficient to show that in any C^*-algebra the spectrum of each self-adjoint element is real. This result holds even if A is not commutative. The proof we give here is due to Richard Arens. It is much easier (exercise) to give a proof for the special case when $A = \mathcal{B}(\mathcal{H})$.

Theorem 17 *Let A be a C^*-algebra with identity I. If a is a self-adjoint element of A, then* $\text{sp}(a) \subseteq \mathbb{R}$.

Proof: It suffices to show that if $\lambda = x + iy \in \text{sp}(a)$, then $y = 0$. We have already seen that $\lambda \in \text{sp}(a)$ if and only if $\bar{\lambda} \in \text{sp}(a^*)$. Since $a = a^*$, we have $\lambda \in \text{sp}(a)$ if and only if $\bar{\lambda} \in \text{sp}(a)$, so we may assume that $y \geq 0$. Since

$$(a + niI) - (x + (y + n)i)I = a - (x + iy)I$$

we see that $x + (y + n)i \in \text{sp}(a + niI)$. It then follows that $x - (y + n)i = \overline{x + (y + n)i} \in \text{sp}((a + ni)^*) = \text{sp}(a - niI)$. But this implies that

$$|x + (y + n)i| \leq \|a + niI\|, \quad \text{and}$$

$$|x - (y + n)i| \leq \|a - niI\| = \|(a + niI)^*\| = \|a + niI\|.$$

Multiplying these inequalities and using the C^*-identity gives

$$\begin{aligned} x^2 + y^2 + 2ny + n^2 &\leq \|a + niI\|^2 = \|(a + niI)(a + niI)^*\| \\ &= \|(a + niI)(a - niI)\| = \|a^2 + n^2 I\| \leq \|a^2\| + n^2. \end{aligned}$$

Thus

$$0 \leq 2yn \leq \|a\|^2 - (x^2 + y^2)$$

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for all positive integers n, and this is clearly impossible unless $y = 0$. □

We return now to the problem of showing that Γ is an isometry. If Γ is to be an isometry, then for all a in A we must have

$$\|a\| = \|\Gamma(a)\|_\infty = \sup\{|\varphi(a)| : \varphi \in \chi(A)\} = \sup\{|\lambda| : \lambda \in \mathrm{sp}(a)\},$$

so we need to show that in any commutative C^*-algebra with identity we always have $\|a\| = \sup\{|\lambda| : \lambda \in \mathrm{sp}(a)\}$. This latter number $\sup\{|\lambda| : \lambda \in \mathrm{sp}(a)\}$ is the radius of the smallest closed disc centered at the origin which contains the spectrum of a.

Exercises

1. Let A be a Banach algebra, and suppose $J : a \to a^*$ is a conjugate-linear map of A into A such that for all a and b in A we have $a^{**} = a$ and $(ab)^* = b^*a^*$. (That is, assume that J is an involution on A, but do *not* assume that J is an isometry.) Suppose moreover that for every a in A we have $\|aa^*\| \geq \|a\|^2$. Show that J is an isometry and that A is a C^*-algebra (i.e, that for all a in A we actually have equality in the condition $\|aa^*\| \geq \|a\|^2$). [This is a kind of automatic continuity result: the condition "$\|aa^*\| \geq \|a\|^2$ for all a" implies that J is isometric, and hence continuous.]

2. Suppose A is a C^*-algebra, and suppose that a non-zero element I of A satisfies $Ia = a = aI$ for every a in A. (That is, assume that I is a multiplicative identity for A, but do *not* suppose that I has norm 1.) Show that the norm of I must be 1.

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3. Show directly (without using the Arens argument) that any self-adjoint element of $\mathcal{B}(\mathcal{H})$ has real spectrum.

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The spectral radius formula

Let A be a Banach algebra with identity. If x is an element of A, then the *spectral radius* of x is the number $r(x)$ given by

$$r(x) = \sup\{|\lambda| : \lambda \in \sup(x)\}.$$

The following result is important because it tells us that the spectral radius is completely determined by the norm. In particular, if B is a closed subalgebra of A, and if B also has an identity, then the spectral radius of any element of B is the same whether we calculate it in B or in A.

Theorem 18 (Spectral Radius Formula) *Let A be a Banach algebra with identity I, and let x be an element of A. Then*

$$r(x) = \inf_{n \geq 1} \|x^n\|^{1/n} = \lim_{n \to +\infty} \|x^n\|^{1/n}.$$

Proof: Recall that for any polynomial p, the spectrum of $p(x)$ is given by the set $\{p(\lambda) : \lambda \in \mathrm{sp}(x)\}$. In particular, if $\lambda \in \mathrm{sp}(x)$, then $\lambda^n \in \mathrm{sp}(x^n)$ for all n. It follows that $|\lambda^n| \leq \|x^n\|$ for all n and for all $\lambda \in \mathrm{sp}(x)$, so that $|\lambda| \leq \|x^n\|^{1/n}$ for all n and all $\lambda \in \mathrm{sp}(x)$. Thus we have

$$\begin{aligned} r(x) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(x)\} &\leq \inf_{n \geq 1} \|x^n\|^{1/n} \leq \varliminf_{n \to +\infty} \|x^n\|^{1/n} \\ &\leq \varlimsup_{n \to +\infty} \|x^n\|^{1/n}. \end{aligned}$$

It suffices therefore for us to show that $\varlimsup_{n \to +\infty} \|x^n\|^{1/n} \leq r(x)$. We may clearly assume that $x \neq 0$.

Now consider the function $z \to z(I - zx)^{-1}$, which is defined on the set $S = \{z \in \mathbb{C} : I - zx \text{ is invertible}\}$. If $|z| < \frac{1}{\|x\|}$, i.e., if $\|zx\| < 1$, then $I - zx$

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is invertible, and the series $\sum_{n=0}^{+\infty} z^{n+1}x^n$ converges to $z(I-zx)^{-1}$. It follows that for each bounded linear functional φ on A and each z with $|z| < \frac{1}{\|x\|}$, we have $\varphi(z(I-zx)^{-1}) = \sum_{n=0}^{+\infty} z^{n+1}\varphi(x^n)$. In particular, the function $g(z) = \varphi(z(I-zx)^{-1})$ is analytic on the disc $|z| < \frac{1}{\|x\|}$, and $g(z) = \sum_{n=0}^{+\infty} z^{n+1}\varphi(x^n)$ is its Taylor expansion in this disc.

Now we saw earlier that the function $\lambda \to \varphi((x-\lambda I)^{-1}) = -\varphi((\lambda I-x)^{-1})$ is analytic on the resolvent set of x, and in particular on the annulus given by $r(x) < |\lambda| < +\infty$. If we now put $z = \frac{1}{\lambda}$, we see that $h(z) = \varphi(z(I-zx)^{-1}) = \varphi\left(\lambda^{-1}\left(I - \frac{x}{\lambda}\right)^{-1}\right) = \varphi((\lambda I - x)^{-1})$ is analytic for $0 < |z| < \frac{1}{r(x)}$ when $r(x) > 0$, and for $0 < |z|$ when $r(x) = 0$. Clearly h has an analytic extension through the origin, since $\varphi(z(I-zx)^{-1}) = z\varphi((I-zx)^{-1}) \to 0 \cdot \varphi(I) = 0$ as $z \to 0$. Thus h is analytic in a disc D (either $|z| < \frac{1}{r(x)}$ or $\mathbb{C}$) that contains the disc $|z| < \frac{1}{\|x\|}$, and $h = g$ on $|z| < \frac{1}{\|x\|}$. It follows that the Taylor expansion for h in D is also given by $\sum_{n=0}^{+\infty} z^{n+1}\varphi(x^n)$. In particular, the radius of convergence of the power series $\sum_{n=0}^{+\infty} z^{n+1}\varphi(x^n)$ is at least as large as $\frac{1}{r(x)}$ when $r(x) > 0$, and is infinite when $r(x) = 0$. In either case this leads to $\overline{\lim}_{n\to+\infty} |\varphi(x^n)|^{1/n} \le r(x)$, and this inequality holds for every bounded linear functional φ on A. The result now follows from the next proposition.
□

Proposition 39 *Let A be a Banach algebra with identity I, and let x be an element of A. Then*

$$\overline{\lim_{n\to+\infty}} \|x^n\|^{1/n} = \sup\Big\{ \overline{\lim_{n\to+\infty}} |\varphi(x^n)|^{1/n} : \varphi \text{ is a bounded linear functional on } A \Big\}.$$

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Proof: For any φ, $\overline{\lim}_{n\to+\infty}|\varphi(x^n)|^{1/n} \le \overline{\lim}_{x\to+\infty}\|\varphi\|^{1/n}\|x^n\|^{1/n} \le \overline{\lim}_{n\to+\infty}\|x^n\|^{1/n}$, so it suffices to show that $\overline{\lim}_{n\to+\infty}\|x^n\|^{1/n} \le \sup_\varphi\{\overline{\lim}_{n\to+\infty}|\varphi(x^n)|^{1/n}\}$. If $0 = \overline{\lim}_{n\to+\infty}\|x^n\|^{1/n}$, then we have nothing to prove, so suppose $0 < \overline{\lim}_{n\to+\infty}\|x^n\|^{1/n}$. Choose t with $0 < t < \overline{\lim}_{n\to+\infty}\|x^n\|^{1/n}$, and let $w = \frac{1}{t}$, so that $\overline{\lim}_{n\to+\infty}\|w^nx^n\|^{1/n} = w\,\overline{\lim}_{n\to+\infty}\|x^n\|^{1/n} > 1$. Choose s with $1 < s < \overline{\lim}_{n\to+\infty}\|w^nx^n\|^{1/n}$. Then $1 < s < \|w^nx^n\|^{1/n}$ for infinitely many n, and for these n we have $1 < s^n < \|w^nx^n\|$. Thus the sequence $\{w^nx^n\}$ is unbounded. By the Uniform Boundedness Principle, there exists a bounded linear functional φ_0 on A such that $\varphi_0(w^nx^n) = w^n\varphi_0(x^n)$ is unbounded. In particular, the series $\sum_{n=0}^{+\infty} w^n\varphi_0(x^n)$ cannot converge. It follows that the power series $\sum_{n=0}^{+\infty} z^n\varphi_0(x^n)$ has radius of convergence less than or equal to w, i.e., that $\overline{\lim}_{n\to+\infty}|\varphi_0(x^n)|^{1/n} \ge \frac{1}{w} = t$. Thus $t \le \sup_\varphi\{\overline{\lim}_{n\to+\infty}|\varphi(x^n)|^{1/n}\}$. Since t is arbitrary in the interval between zero and $\overline{\lim}_{n\to+\infty}\|x^n\|^{1/n}$, the result is established. □

Exercises

1. Give an example of an operator T on a Hilbert space H for which the spectral radius $r(T)$ is strictly less than $\|T\|$.

2. Let k be any strictly positive real number. Give an example of a two by two matrix with spectrum $\{0\}$ and norm greater than or equal to k.

3. Let $\{a\}$ be a sequence which converges to 0, and let T be the weighted shift $Tx = \sum_{n=1}^{\infty} a_{n+1}\langle x, e_n\rangle e_{n+1}$. What is the spectral radius of T? (This spectral radius can be computed directly, but can be deduced

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quickly from the spectral properties of compact operators to be developed later.)

4. Let a be an element of a Banach algebra A with identity I, and let ρ be the spectral radius of a. Show that the following conditions are equivalent.

   a) $\rho < 1$

   b) there exists a positive integer k with $\|a^k\| < 1$

   c) the sequence $\|a^k\|$ converges to zero

   d) the series $\sum_{i=0}^{+\infty} a^k$ converges in A (with respect to the norm topology).

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The little Gelfand-Naimark Theorem

Theorem 19 (Little Gelfand-Naimark Theorem) *Let A be a commutative C^*-algebra with identity. Then the Gelfand transform Γ is an isometric $*$-isomorphism of A onto $C(\chi(A))$.*

Proof: Since A is a C^*-algebra, every character of A is real on self-adjoint elements of A, so Γ is a norm-decreasing $*$-homomorphism of A onto a dense $*$-subalgebra of $C(\chi(A))$. Let a be self-adjoint. Then $\|a^2\| = \|a^*a\| = \|a\|^2$ by the C^*-algebra identity. Moreover, for any positive integer n, we have $\|a^{2^{n+1}}\| = \|a^{2^n}a^{2^n}\| = \|a^{2^n}(a^{2^n})^*\| = \|a^{2^n}\|^2$, where we have again used the C^*-identity to obtain the third equation in this sequence. By an easy induction argument, we have $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \geq 1$, so that $\|a^{2^n}\|^{1/2^n} = \|a\|$ for all such n. It then follows from the spectral radius formula that

$$\|a\| = \sup\{|\lambda| : \lambda \in \mathrm{sp}(a)\} = \sup\{|\varphi(a)| : \varphi \in \chi(a)\} = \|\Gamma(a)\|_\infty.$$

Thus Γ preserves the norms of self-adjoint elements. To get the general case, observe that for any a, a^*a is self-adjoint, so by the previous case,

$$\|\Gamma(a)\|_\infty^2 = \|\overline{\Gamma(a)}\Gamma(a)\|_\infty = \|\Gamma(a^*a)\|_\infty = \|a^*a\| = \|a\|^2.$$

Thus Γ is isometric on A, so its image $\Gamma(A)$ is complete, and hence is $\Gamma(A)$ closed in $C(\chi(A))$. Since $\Gamma(A)$ is dense, Γ maps onto $C(\chi(A))$. Every isometric map is certainly injective, so Γ is an isometric $*$-isomorphism of A onto $C(\chi(A))$, as claimed. □

Corollary 23 *If A is a C^*-algebra with identity, then A is commutative if and only if there exists a compact Hausdorff space X such that A is isometrically $*$-isomorphic to $C(X)$.*

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If X and Y are compact Hausdorff spaces and X is homeomorphic to Y, then it is easy to see (exercise) that $C(X)$ and $C(Y)$ are isometrically $*$-isomorphic. In the exercises you will also show that if $C(X)$ and $C(Y)$ are isomorphic, then X and Y must be homeomorphic. Thus the compact Hausdorff space of the last corollary is unique up to homeomorphism. The space X can be naturally identified (see the exercises) with the character space of $C(X)$.

Observe that if S is any subset of a C^*-algebra A, then the intersection of all the C^*-subalgebras of A which contain S is again a C^*-subalgebra containing S, and hence is the smallest such. This C^*-subalgebra is called the C^*-subalgebra of A *generated* by S.

Suppose now that A is a C^*-algebra with identity I, and let a be an element of A. When a is normal, then the set of all polynomials in a and a^* is a commutative $*$-subalgebra of A. Now (exercise) the closure of any $*$-subalgebra of A is again a $*$-subalgebra of A, and the closure of a commutative algebra is commutative. It follows that when a is normal, the C^*-algebra generated by a and the identity is precisely the closure of the set of all polynomials in a and a^*, and in particular is commutative. Conversely, if the C^*-algebra generated by a and I is commutative, then a commutes with a^*, so a is normal. We shall use the notation $C^*(a)$ to stand for the C^*-algebra generated by a and I. Thus $C^*(a)$ is commutative if and only if a is normal.

Proposition 40 *If a is a normal element of a C^*-algebra A with identity, then the spectral radius of a coincides with its norm.*

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Proof: Since the norm on $C^*(a)$ is the norm inherited from A, it follows immediately from the spectral radius formula that the spectral radius of a in A coincides with the spectral radius of a in $C^*(a)$. Thus it suffices to assume that $A = C^*(a)$, and hence that A is commutative. Since the Gelfand transform Γ is an isomorphism of A onto $C(\chi(A))$, the spectrum of a in A coincides with the spectrum of $\Gamma(a)$ in $C(\chi(A))$, and hence the spectral radius of a in A coincides with the spectral radius of $\Gamma(a)$ in $C(\chi(A))$. But Γ is also isometric, so it suffices to check that the norm and the spectral radius coincide on each element of $C(\chi(A))$. In any algebra of the form $C(X)$, the spectrum of f is just its range, so the spectral radius of f agrees with the supremum norm $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ of f. Taking $X = \chi(A)$ establishes the result. $\square$

Exercises

1. Let A be a normed $*$-algebra.

    a. Show that the norm closure of any $*$-subalgebra of A is again a $*$-subalgebra of A.

    b. Show that if B is a commutative subalgebra of A, then the norm closure of B is also commutative subalgebra of A.

2. If a is normal, then clearly the powers of a commute with the powers of a^*. Use this fact, together with the Spectral Radius Formula, to give a direct proof (not quoting the little Gelfand-Naimark Theorem) that for normal element a of any C^*-algebra with identity, the spectral radius and the norm of a coincide.

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3. [**The characters and maximal ideals of** $C(X)$] Let X be a compact Hausdorff space, and let $C(X)$ be the algebra of all continuous complex-valued functions on X.

   a. Show that for each x in X, the function ϵ_x, defined by $\epsilon_x(f) = f(x)$, is a character of $C(X)$.

   b. Let J be a proper ideal of $C(X)$. Show that there exists an element x_J of X such that $f(x_J) = 0$ for all f in J. [Suggestion: Suppose not; use the fact that J is an ideal to show that for each x in X there exists a function g_x in J such that $g_x(x) > 0$; conclude from compactness that there exists g in J with $g(x) > 0$ for all x in X.]

   c. Show that if J is a maximal ideal, then the point x_J of the previous part is unique and that $J = \{f \in C(X) : f(x_J) = 0\}$. [You may want to use Urysohn's Lemma, the Tietze Extension Theorem or some similar result from topology.] Conclude that every character of $C(X)$ is of the form ϵ_x, where $x \in X$.

   d. Show that the function $x \to \epsilon_x$ is a homeomorphism of X onto the character space (with the weak* topology) of $C(X)$.

4. [**a little more challenging**] With X and $C(X)$ as in the last exercise, show that the function $S \to \{f \in C(X) : f = 0 \text{ on } S\}$ is an order-reversing bijection from the set of all closed subsets of X onto the set of all closed ideals in $C(X)$.

5. If X and Y are compact Hausdorff spaces and π is a homeomorphism of X onto Y, then the map $\pi^* : f \to f \circ \pi$ is an isometric $*$-isomorphism of $C(Y)$ onto $C(X)$.

6. Show that if A and B are commutative C^*-algebras with identities, and

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if $\Phi : A \to B$ is an algebra isomorphism (onto), then $\Phi^* : \psi \to \psi \circ \Phi$ is a homeomorphism of the character space of B onto the character space of A. In particular, if $\Phi : C(X) \to C(Y)$ is an algebra isomorphism (onto), then Y is homeomorphic to X. Use this to show that if Φ is an algebra homomorphism of $C(X)$ onto $C(Y)$, then Φ is also a $*$-homomorphism.

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Spectral permanence for C^*-algebras

We are almost ready to prove our first version of the Spectral Theorem. To do so, we need one more important property of C^*-algebras. Recall that if A is an algebra with identity I and B is a subalgebra of A containing I, then for each b in B we have $\mathrm{sp}_A(b) \subseteq \mathrm{sp}_B(b)$. For C^*-algebras and C^*-subalgebras, this containment must in fact be an equality. This property is called *spectral permanence.*

Proposition 41 *Let A be a C^*-algebra with identity I, and suppose $a \in A$ has an inverse a^{-1} in A. Then $a^{-1} \in C^*(a)$.*

Proof: Suppose first that $a = a^*$. Let B be the C^*-algebra generated by a and a^{-1}. Then B is the closure of the set of all polynomials in a and a^{-1}, so B is commutative. Let Γ be the Gelfand transform of B, and let $K = \mathrm{sp}_B(a)$ be the spectrum of a with respect to the algebra B. Since a is self-adjoint and B is a C^*-algebra, $K \subseteq \mathbb{R}$. Since a is invertible in B, K is a compact set that does not contain zero. Thus the function $h(\lambda) = \frac{1}{\lambda}$ is continuous from K to $\mathbb{R}$. By the Weierstrass Approximation Theorem, there is a sequence $\{p_n\}$ of polynomials (in one real variable and with real coefficients) that converges uniformly on K to h. Now for any polynomial $p(\lambda) = c_0 I + c_1\lambda + \cdots + c_k\lambda^k$ and any character φ of B, we have

$$\varphi(p(a)) = \varphi(c_0 I + c_1 a + \cdots + c_k a^k) = c_0 + c_1\varphi(a) + \cdots + c_k(\varphi(a))^k = p(\varphi(a)),$$

and $\varphi(a) \in K$ for all characters φ of B. It follows that for all characters φ of B,

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$$\begin{aligned}[\Gamma(p_n(a)](\varphi) = \varphi(p_n(a)) = p_n(\varphi(a)) \to h(\varphi(a)) \\ = \frac{1}{\varphi(a)} = \varphi(a^{-1}) = [\Gamma(a^{-1})](\varphi),\end{aligned}$$

and that the convergence is uniform in φ, since $p_n \to h$ uniformly on K. Thus

$$\begin{aligned}\|\Gamma(p_n(a) - a^{-1})\|_\infty &= \|\Gamma(p_n(a)) - \Gamma(a^{-1})\|_\infty \\ &= \sup\{|[\Gamma(p_n(a)) - \Gamma(a^{-1})](\varphi)| : \varphi \in \chi(B)\} \to 0.\end{aligned}$$

But the Gelfand transform of a C^*-algebra is isometric, so

$$\|p_n(a) - a^{-1}\| \to 0$$

in B. Clearly each $p_n(a)$ lies in $C^*(a)$, so a^{-1} also lies in $C^*(a)$.

For the general case, suppose again that a has an inverse a^{-1} in A. Then a^* is invertible in A, so aa^* is self-adjoint and invertible in A, with inverse equal to $(a^*)^{-1}a^{-1}$. By the previous case $(aa^*)^{-1}$ lies in $C^*(aa^*)$. But $aa^* \in C^*(a)$, so $C^*(aa^*) \subseteq C^*(a)$, so $(aa^*)^{-1} \in C^*(a)$. But then $a^{-1} = a^*(a^*)^{-1}a^{-1} = a^*(aa^*)^{-1} \in C^*(a)$ as required. □

Corollary 24 (Spectral Permanence) *Let A be a C^*-algebra with identity I, and let B be a C^*-subalgebra of A which contains I. Then for any b in B, b is invertible in B if and only if b is invertible in A. Consequently, for each b in B, the spectra of b in A and in B coincide.*

Proof: Let b be any element of B. Then $C^*(b)$ is a C^*-subalgebra of B. From the proposition, applied to B and $C^*(b)$, we see that b is invertible in B if and only if b is invertible in $C^*(b)$. From the proposition applied to A and

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$C^*(b)$ we see that b is invertible in A if and only if b is invertible in $C^*(b)$. Thus the first assertion is established. The second follows immediately from the first. $\square$

Exercises

1. Give an example to show that if A is a C^*-subalgebra of B with a *different* identity, then spectral permanence need not hold. That is, give an example of a C^*-algebra B with identity I, a C^*-subalgebra A of B with identity $E \neq I$, and an element a of A whose spectrum in B is properly contained in its spectrum in A.

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The Spectral Theorem and the Continuous Functional Calculus

We now have all the tools we need to prove our first version of the Spectral Theorem. This version says that for any normal operator T, the C^*-algebra generated by T and I is naturally $*$-isomorphic and isometric to the algebra $C(\mathrm{sp}(T))$. This natural $*$-isomorphism identifies T with the identity function on $\mathrm{sp}(T)$, and is the unique $*$-isomorphism that makes this particular identification. Moreover, its inverse carries the algebraic, order, and metrical properties of $C(\mathrm{sp}(T))$ over to corresponding properties in $C^*(T)$. This has a myriad of useful consequences, such as the fact that each positive operator T has a unique positive square root, or the fact that $C^*(T)$ contains a non-trivial projection (i.e., a projection different from 0 and I) if and only if $\mathrm{sp}(T)$ is not a connected topological space. (See the next few sections for these and other applications.) Many of these consequences are established just by showing that they hold in $C(\mathrm{sp}(T))$ and invoking the existence of the isometric $*$-isomorphism. Since the isomorphism is a one-to-one correspondence between the elements of $C^*(T)$ and the functions which comprise $C(\mathrm{sp}(T))$, it is called the *continuous functional calculus.* Note that by Spectral Permanence, the spectrum of T is the same set, whether we regard T as an element of $\mathcal{B}(\mathcal{H})$ or of $C^*(T)$ (or of any C^*-algebra in between).

It is no more difficult to prove this first version of the Spectral Theorem for an arbitrary normal element of a C^*-algebra with identity than it is for an arbitrary normal operator in $\mathcal{B}(\mathcal{H})$, so we shall prove the more general case. (Actually, the greater generality is only apparent, since the Gelfand-

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Naimark Theorem shows that every C^*-algebra with identity is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity of $\mathcal{B}(\mathcal{H})$.)

Spectral Theorem (The Continuous Functional Calculus) *Let a be a normal element of a C^*-algebra with identity I. Let $A = C^*(a)$, and let Γ be the Gelfand transform of A. Then*

a. $\Gamma(a)$ *is a homeomorphism from the character space* $\chi(A)$ *onto* $\mathrm{sp}(a)$*,*

b. for each f *in* $C(\mathrm{sp}(a))$ *there exists a unique element* a_f *of* A *such that* $\Gamma(a_f) = f \circ \Gamma(a)$*, and*

c. the map $\rho : f \longrightarrow a_f$ *is an isometric* $*$*-isomorphism of* $C(\mathrm{sp}(a))$ *onto* A*.*

Moreover, ρ takes the constant function 1 in $C(\mathrm{sp}(a))$ to the identity I of A, ρ takes the identity function $id(\lambda) = \lambda$ in $C(\mathrm{sp}(a))$ to the generator a of $C^*(a)$, and ρ takes complex conjugation on $sp(a)$ to the element a^* of $C^*(a)$.

Proof: (See Fig. 1.) We have already seen that $\Gamma(a)$ is continuous, and that the range of $\Gamma(a)$ is $\mathrm{sp}(a)$. If two characters agree on a, then (since characters are continuous $*$-homomorphisms) they must agree on every polynomial in a and a^*, and hence on the closure $C^*(a)$ of the polynomials in a and a^*. It follows that $\Gamma(a)$ is one-to-one. Since $\chi(A)$ is compact and $\mathrm{sp}(a)$ is Hausdorff, $\Gamma(a)$ is a homeomorphism.

Now $f \circ \Gamma(a)$ is a composition of continuous functions, and so is continuous from $\chi(A)$ into $\mathbb{C}$. Thus there is a unique a_f in A such that $\Gamma(a_f) = f \circ \Gamma(a)$, since Γ maps A onto $C(\chi(A))$. (In fact, $a_f = \Gamma^{-1}(f \circ \Gamma(a))$.)

Next, we have $\Gamma(a) = id \circ \Gamma(a)$, so by uniqueness of a_{id}, $a = a_{id}$, i.e., $\rho(id) = a$. Similarly, $\Gamma(a^*) = \overline{\Gamma(a)} =$ (complex conjugation) $\circ\, \Gamma(a)$ and

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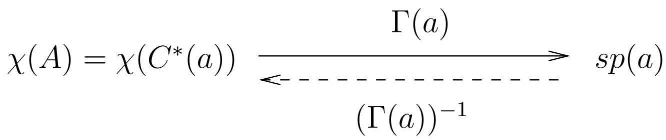
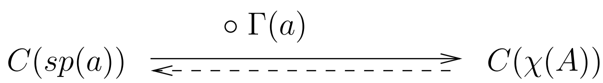
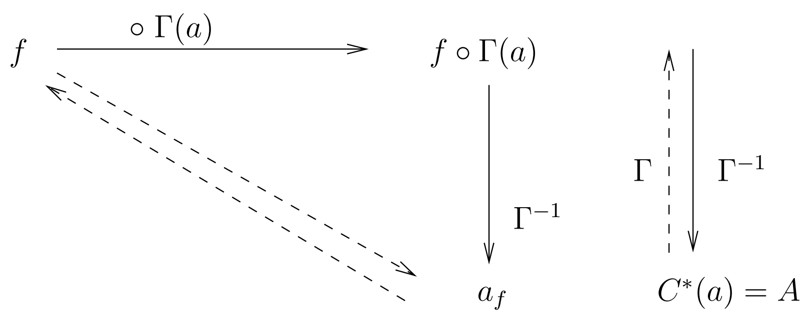
The homeomorphism of $\chi(C^*(a))$ onto $sp(a)$:



The isometric $*$-isomorphism of $C(sp(a))$ onto $A = C^*(a)$:





$\Gamma(a_f) = f \circ \Gamma(a)$, i.e., $a_f = \Gamma^{-1}(f \circ \Gamma(a))$, i.e.,

$$\boxed{f(a) = \Gamma^{-1}(f \circ \Gamma(a))}$$

Figure 1: The mappings of the Continuous Functional Calculus

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uniqueness imply that ρ takes complex conjugation to a^*. Since $\Gamma(a)$ is a homeomorphism, it is easy to check that $\rho : f \to a_f = \Gamma^{-1}(f \circ \Gamma(a))$ is an isometric $*$-isomorphism onto A. Clearly $\rho(1) = I$, since ρ is a surjective ring homomorphism. □

Remark 9 The elements a and id are generators respectively for the C^*-algebras $C^*(a)$ and $C(\mathrm{sp}(a))$ respectively. It follows immediately that ρ is the only isometric $*$-isomorphism between these algebras that takes id to a. Since $*$-isomorphisms of C^*-algebras are automatically isometric (we shall establish this in a later chapter), ρ is in fact the only $*$-isomorphism between these algebras that takes id to a.

Notation Since the algebraic and metrical properties of a_f must reflect those of f, it is customary to write $f(a)$ for a_f. If p is a polynomial in one variable, then since Γ is a $*$-isomorphism, we have $\Gamma(p(a)) = p(\Gamma(a)) = p \circ \Gamma(a)$ in $C(\chi(A))$. (This just means that for all characters of A we have $\varphi(p(a)) = p(\varphi(a))$.) In particular, this notation ("$p(a)$" for "a_p") is consistent with our earlier use of $p(a)$ when p is a polynomial. More generally, if p is a polynomial in two variables, and if $h(z) = p(z, \bar{z})$, then $\Gamma(p(a, a^*)) = p(\Gamma(a), \overline{\Gamma(a)}) = h \circ \Gamma(a)$ in $C(\chi(A))$, so that $a_h = p(a, a^*)$.

Corollary 25 *Let $p_n(z, \bar{z})$ be a sequence of polynomials in z and $\bar{z}$, and suppose that $h_n(z) = p_n(z, \bar{z})$ converges uniformly on* $\mathrm{sp}(a)$ *to a function h. Then $h_n(a) = p_n(a, a^*)$ converges to $h(a)$ in $C^*(a)$. In particular if h is analytic in an open disc D centered at zero which contains* $\mathrm{sp}(a)$*, then the series $\sum_{n=0}^{\infty} \frac{h^n(0)}{n!} a^n$ converges to $h(a)$ in $C^*(a)$.*

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Proof: Since $h_n - h \to 0$ uniformly on $\mathrm{sp}(a)$, we have $\|h_n(a) - h(a)\| = \|h_n - h\|_\infty \to 0$, so $h_n(a)$ converges to $h(a)$ in $C^*(a)$. If h is analytic in D, then the partial sums of the Maclauren series for h converge uniformly to h on every compact subset of D, so the second assertion follows immediately upon application of the first assertion to this sequence of partial sums. □

Thus for any normal a we have norm convergent expansions $a = \sum_{n=0}^{\infty} \frac{1}{n!} a^n$, $\cos a = I - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \cdots$, etc.

Corollary 26 (The Spectral Mapping Theorem) *Let a be a normal element of a C^*-algebra A with identity I. Let $f \in C(\mathrm{sp}(a))$. Then*

$$\mathrm{sp}(f(a)) = f(\mathrm{sp}(a)) = \{f(\lambda) : \lambda \in \mathrm{sp}(a)\}.$$

(That is, $sp(f(a)) = \mathrm{Range}(f)$.)

Proof: Since the function $\rho : f \to f(a)$ is a $*$-isomorphism of $C(sp(a))$ onto $C^*(a)$, we have $sp(f(a)) = sp_{C^*(a)}(\rho(f)) = sp_{C(sp(a))}(f) = \mathrm{Range}(f) = \{f(\lambda) = \lambda \in sp(a)\}$. □

The Spectral Mapping Theorem will be used often in the sequel. As a typical illustration of its use we cite the following argument. If a is self-adjoint, then the spectrum of a is real, so e^{ia} is a normal element of $C^*(a)$ with spectrum $\{e^{it} : t \in sp(a)\}$ contained in the unit circle. Now an element of $C(X)$ is unitary if and only if its spectrum (i.e., range) is contained in the unit circle, so the same is true of the elements of $C^*(a)$. Thus we see that for a self-adjoint a, e^{ia} is always unitary.

For the sake of completeness, we restate the results of this section for normal operators on a Hilbert space. These restatements are of course special cases and hence require no further proof.

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Spectral Theorem (The Continuous Functional Calculus) *Let T be a normal operator in $\mathcal{B}(\mathcal{H})$. Let $A = C^*(T)$, and let Γ be the Gelfand transform of A. Then*

a. *$\Gamma(T)$ is a homeomorphism from the character space $\chi(A)$ onto $\mathrm{sp}(T)$,*

b. *for each f in $C(\mathrm{sp}(T))$ there exists a unique element T_f of A such that $\Gamma(T_f) = f \circ \Gamma(T)$, and*

c. *the map $\rho : f \to T_f$ is an isometric $*$-isomorphism of $C(\mathrm{sp}(T))$ onto A.*

Moreover, ρ takes the constant function 1 in $C(\mathrm{sp}(T))$ to the identity I of $\mathcal{B}(\mathcal{H})$, ρ takes the identity function $id(\lambda) = \lambda$ in $C(\mathrm{sp}(T))$ to the operator T, and ρ takes complex conjugation on $\mathrm{sp}(T)$ to the operator T^*.

Corollary 27 (The Spectral Mapping Theorem) *Let T be a normal operator in $\mathcal{B}(\mathcal{H})$. Let $f \in C(\mathrm{sp}(T))$. Then*

$$\mathrm{sp}(f(T)) = f(\mathrm{sp}(T)) = \{f(\lambda) : \lambda \in \mathrm{sp}(T)\}.$$

Exercises

1. Let a be a normal element of a C^*-algebra with identity I, and let s and t be scalars. What is the spectrum of $sa + tI$, and why? (There are several reasonable answers to "why?")

2. Let a be a normal element of a C^*-algebra with identity I, and suppose $sp(a) \subseteq (-1, 1)$.

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a) Show that $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} a^n$ converges in norm to $\log(I + A)$.

b) Use Abel's Theorem (see for example p. 325 of R. G. Bartle's, *The Elements of Real Analysis*) to show that if $sp(a) \subseteq [0, 1]$, then $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} a^n$ converges in norm to $\log(I + A)$.

3. Show that a normal element a of a C^*-alegbra with identity is the square of some self-adjoint b in $C^*(a)$ if and only if $sp(a) \subseteq [0, +\infty)$, i.e., if and only if a is positive.

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Some applications of the spectral theorem

In the next few sections we shall use the Spectral Theorem to obtain information about normal operators. Many of the results will also hold, with the same proofs, in a general C^*-algebra with identity, and we will establish them in the general context wherever practicable.

Theorem 20 *Let a be a normal element of a C^*-algebra A with identity I. Then*

a. a is self-adjoint if and only if $\mathrm{sp}(a)$ is real, and

b. a is unitary if and only if the spectrum of a is contained in the unit circle T.

Proof: By Spectral Permanence we may assume that $A = C^*(a)$. Since a is normal, we may assume that A is commutative. Let Φ be any $*$-isomorphism of A onto $C(X)$, e.g., the Gelfand transform of A. Then

$$a = a^* \text{ if and only if } \Phi(a) = \overline{\Phi(a)} \text{ if and only if } \Phi(a) \text{ is real-valued, and}$$
$$aa^* = I = a^*a \text{ if and only if}$$
$$\Phi(a)\overline{\Phi(a)} = 1 = \overline{\Phi(a)}\Phi(a) \text{ if and only if } \mathrm{Range}(\Phi(a)) \subseteq T.$$

But $\mathrm{sp}(a) = sp(\Phi(a)) = \mathrm{Range}(\Phi(a))$, so the result follows. □

In particular, we get the useful fact (see the previous section) that for self-adjoint a, e^{ia} is always unitary.

Theorem 21 *Let T be an element of $\mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:*

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a. $\langle Tx, x\rangle \geq 0$ *for all* x *in* $\mathcal{H}$*;*

b. there exists S *in* $\mathcal{B}(\mathcal{H})$ *with* $T = S^*S$*;*

c. there exists a self-adjoint S *in* $\mathcal{B}(\mathcal{H})$ *with* $T = S^2$*;*

d. T *is normal and* $\mathrm{sp}(T) \subseteq [0, +\infty)$.

Moreover, for an element T *of a* C^**-algebra* A *with identity, if we replace* $\mathcal{B}(\mathcal{H})$ *by* A*, then we have d.* $\Rightarrow$ *c.* $\Rightarrow$ *b.*

Proof: d. $\Rightarrow$ c.: Since T is normal, $C(sp(T))$ is $*$-isomorphic to $C^*(T)$ under a map ρ that takes the identity function on $\mathrm{sp}(T)$ to T. The range of this identity function is $\mathrm{sp}(T)$ and $\mathrm{sp}(T) \subseteq [0, +\infty)$, so this identity function is non-negative on $\mathrm{sp}(T)$. Thus it has a unique non-negative square root $\sqrt{} : x \to \sqrt{x}$ in $C(\mathrm{sp}(T))$. Since $\sqrt{}$ is real-valued, it is self-adjoint in $C(\mathrm{sp}(T))$. If we now put $S = \rho(\sqrt{})$, then S is a self-adjoint element of $C^*(T)$ such that $S^2 = T$.

c. $\Rightarrow$ b.: trivial

b. $\Rightarrow$ a.: obvious

a. $\Rightarrow$ d.: $\langle Tx, x\rangle \geq 0$ for all x in $\mathcal{H}$, so $\langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} = \langle Tx, x\rangle$ for all x, so $T = T^*$ and $\mathrm{sp}(T) \subseteq \mathbb{R}$. Let $\lambda \in \mathrm{sp}(T)$. Since T is normal, its residual spectrum is empty, and its continuous spectrum is contained in its approximate point spectrum. It follows that we may choose a sequence of unit vectors x_n in $\mathcal{H}$ such that $\|(T - \lambda I)x_n\| < \frac{1}{n}$. Then $|\langle Tx_n, x_n\rangle - \lambda| = |\langle (T - \lambda I)x_n, x_n\rangle| \leq \|(T - \lambda I)x_n\|\|x_n\| \to 0$, so $0 \leq \langle Tx_n, x_n\rangle \to \lambda$. Thus $\lambda \geq 0$ for all elements λ of $\mathrm{sp}(T)$.

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All of these arguments except the ones involving condition a. are valid in any C^*-algebra with identity, so the theorem is established. □

Corollary 28 *Let $\mathcal{B}(\mathcal{H})_+$ be the collection of all positive operators in $\mathcal{B}(\mathcal{H})$. Then*

a. $\mathcal{B}(\mathcal{H})_+$ is closed under addition and under multiplication by a non-negative scalar (i.e., $\mathcal{B}(\mathcal{H})_+$ is a cone in $\mathcal{B}(\mathcal{H})$).

b. The relation $\geq$ is a partial order on $\mathcal{B}(\mathcal{H})_+$.

c. If $T \geq S$ in $\mathcal{B}(\mathcal{H})$ then $C^*TC \geq C^*SC$ holds for all $C \in \mathcal{B}(\mathcal{H})$. In particular, if T is positive, then C^*TC is positive for every C in $\mathcal{B}(\mathcal{H})$.

d. $\mathcal{B}(\mathcal{H})_+$ is norm-closed in $\mathcal{B}(\mathcal{H})$.

Proof: Exercise. □

Definition 22 An element T of a C^*-algebra with identity is *positive* if it satisfies condition d. of the last theorem. We write $T \geq 0$ to indicate that T is positive, and $T \geq S$ to indicate that $T - S$ is positive.

By the theorem, this use of "positive" agrees with our earlier use of the word in the case of elements of $\mathcal{B}(\mathcal{H})$, and positive elements of $\mathcal{B}(\mathcal{H})$ satisfy all four of the conditions in the theorem.

Remark 10 It can be shown that conditions b. through d. of the last theorem are actually equivalent for elements of any C^*-algebra A with identity, but the proof is lengthy. It proceeds by constructing a Hilbert space $\mathcal{H}$ such that A is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, after which

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one has the equivalence of conditions a) through d) in this $\mathcal{B}(\mathcal{H})$. As a by-product of this construction, we get that the corollary above holds for the positive cone of any C^*-algebra A with identity. The process of obtaining $\mathcal{H}$ from A is called the *Gelfand-Naimark-Segal Construction*, and the embedding of A in $\mathcal{B}(\mathcal{H})$ is the *Gelfand-Naimark Theorem*.

Thus a normal element of a C^*-algebra with identity is self-adjoint, unitary or positive if and only if its spectrum lies in $\mathbb{R}$, in the unit circle $\mathbf{T}$, or in $[0, +\infty)$ respectively. It follows that the self-adjoint, unitary and positive elements of $C(X)$ are the functions whose ranges are contained in $\mathbb{R}$, $\mathbf{T}$, and $[0, +\infty)$ respectively. In particular, note that a non-negative element of $C(X)$ has exactly one non-negative square root in $C(X)$, and hence that a positive element of a commutative C^*-algebra with identity has exactly one positive square root in that algebra. In fact, this existence and uniqueness holds for any C^*-algebra with identity, commutative or not, as the following result shows.

Theorem 22 *Let T be a positive element of a C^*-algebra A with identity I. Then T has a unique positive square root in A.*

Proof: As in the proof of the last theorem, we see that there exists an element S of $C^*(T)$, corresponding under the $*$-isomorphism ρ with the non-negative square root function $\sqrt{}$ on $\mathrm{sp}(T)$, such that $S^* = S$ and $S^2 = T$. According to our notation for the elements of $C^*(T)$, we have $S = \sqrt{T}$, and by the Spectral Mapping Theorem we have $\mathrm{sp}(S) = \left\{\sqrt{\lambda} : \lambda \in \mathrm{sp}(T)\right\}$, so $\mathrm{sp}(S) \subseteq [0, +\infty)$. Thus S is a positive element of $C^*(T)$. Clearly S is the unique such root in $C^*(T)$. It remains only to show that S is the unique

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positive square root of T in A. To see this, suppose V is also a positive square root of T in A. Then V is normal, so $C^*(V)$ is commutative, and $T = V^2$ lies in $C^*(V)$. But then $S = \sqrt{T} \in C^*(T) \subseteq C^*(V)$, so that S and V are positive square roots of T in the commutative algebra $C^*(V)$. By the uniqueness in the commutative case, we have $V = S$. □

The reader is invited to check that there is an analogous result for nth roots, proved in the same way.

Every real-valued continuous function on X is the difference of two non-negative continuous functions on X. This property is also reflected in the functional calculus.

Theorem 23 *Let a be a self-adjoint element of a C^*-algebra with identity. Then there exist unique positive elements a_+ and a_- of $C^*(a)$ such that $a = a_+ - a_-$ and $a_+ a_- = 0$.*

Proof: Let Φ be any isometric $*$-isomorphism of $C^*(a)$ onto an algebra of the form $C(X)$. We identify a with a real-valued function f in $C(X)$. The pointwise maximum and minimum of two functions in $C(X)$ is again a continuous function, so

$$f_+(x) = \max\{0, f(x)\} \quad \text{and} \quad f_-(x) = -\min\{0, f(x)\}$$

are non-negative functions in $C(X)$ with product zero and difference f. It follows that the corresponding elements a_+ and a_- of $C^*(a)$ have the required properties. The uniqueness in $C^*(a)$ follows from the uniqueness of f_+ and f_- in $C(X)$, which is easy to establish. □

The elements a_+ and a_- of the last theorem are called the *positive* and *negative parts* of a respectively.

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Corollary 29 *Every element of a C^*-algebra A with identity is a linear combination of four positive elements of A.*

The next few results are useful in finding non-trivial projections and eigenvalues and eigenspaces.

Proposition 42 *Let a be a normal element of a C^*-algebra A with identity I. Then $C^*(a)$ contains a projection different from 0 and I if and only if $\mathrm{sp}(a)$ is disconnected. If $A = C(X)$, then the projections of A are precisely the characteristic functions of the subsets of X which are both open and closed.*

Proof: Exercise. □

Theorem 24 *Let T be a normal operator, and let λ_0 be an isolated point of $\mathrm{sp}(T)$. Then λ_0 is an eigenvalue of T, and the eigenspace of λ_0 is the range of the projection corresponding to the characteristic function of $\{\lambda_0\}$.*

Proof: Let f be the characteristic function of the singleton set $\{\lambda_0\}$. Since λ_0 is isolated in $\mathrm{sp}(T)$, $\{\lambda_0\}$ is open and closed, so f is a non-zero projection in $C(\mathrm{sp}(T))$. By the spectral theorem, $f(T)$ is a non-zero projection in $C^*(T)$.

Now $(\lambda - \lambda_0 1)f(\lambda) = 0$ for all λ in $\mathrm{sp}(T)$, so $(T - \lambda_0 I)f(T)$ is the zero operator. Since $f(T)$ is not zero, we may choose a non-zero $y = f(T)x$ in the range of $f(T)$. Then

$$(T - \lambda_0 I)y = (T - \lambda_0 I)f(T)x = 0,$$

so λ_0 is an eigenvalue for T with eigenvector $y = f(T)x$. In particular, we see that every non-zero vector in the range of $f(T)$ is an eigenvector for λ_0 and T, so the eigenspace of λ_0 contains the range of $f(T)$.

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On the other hand, suppose $Tx - \lambda_0 x = 0$. Let $g(\lambda) = 1 - f(\lambda)$, and let $h(\lambda) = \frac{1}{\lambda - \lambda_0}$ if $\lambda \neq \lambda_0$ and $h(\lambda_0) = 0$. Then, since λ_0 is isolated in $\mathrm{sp}(T)$, g and h are continuous on $\mathrm{sp}(T)$. For all λ in $\mathrm{sp}(T)$ we have $(\lambda - \lambda_0)h(\lambda) = g(\lambda) = h(\lambda)(\lambda - \lambda_0)$, so that

$$(T - \lambda_0 I)h(T) = g(T) = h(T)(T - \lambda_0 I).$$

Thus $g(T)x = h(T)(T - \lambda_0 I)x = 0$. But $g + f = 1$ on $\mathrm{sp}(T)$, so $g(T) + f(T) = I$, so

$$f(T)x = g(T)x + f(T)x = x.$$

In particular, every eigenvector x for λ_0 and T lies in the range of $f(T)$. $\square$

When we have established the measurable functional calculus (see below), we will be able to significantly sharpen this last result.

Exercises

1. Prove Corollary 28.

2. Prove Proposition 42.

3. Let A be a C^*-algebra with identity, and let $a = a^*$ in A. Suppose a_1 and a_2 are positive elements of A such that $a = a_1 - a_2$ and $a_1 a_2 = 0$. Show that $a_1 = a_+$ and $a_2 = a_-$ (with the notation as in Theorem 23). That is, show that the decomposition guaranteed by Theorem 23 in the algebra $C^*(a)$ is the unique such decomposition in A (as well as the unique one in $C^*(a)$). [Hint: Consider the C^*-algebra generated by a_1, a_2 and the identity of A.]

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4. Let A be a C^*-algebra with identity, and let a be an element of A.

   a) Show that if a is self-adjoint and $\|a\| \leq 1$, then a is the average of two unitary elements of $C^*(a)$.

   b) Show that a is a linear combination of four unitary elements of A.

   c) Show that if an element b of A commutes with every unitary in A, then b commutes with *every* element of A (i.e., b lies in the center of A).

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Some operator monotone functions

In this section we use the continuous functional calculus to show that inversion and the square root function respect the order relation among the positive elements of $\mathcal{B}(\mathcal{H})$. We will use these results extensively when we take up the study of operator means in Part II.

Definition 23 A continuous real-valued function f on an interval J is *operator monotone increasing* on J if it has the property that $A \leq B$ implies $f(A) \leq f(B)$ for all self-adjoint A and B in $\mathcal{B}(\mathcal{H})$ with spectra contained in J. Similarly f is *operator monotone decreasing* on J if $A \leq B$ implies $f(A) \geq f(B)$ for all self-adjoint A and B in $\mathcal{B}(\mathcal{H})$ with spectra contained in J.

Lemma 4 *Let A be a positive element of a C^*-algebra with identity I. Then $A \leq I$ if and only if $\|A\| \leq 1$. If A is positive and invertible, then $A \leq I$ if and only if $A^{-1} \geq I$.*

Proof: From the functional calculus, it suffices to check this for a non-negative function in $C(X)$, and there the result is clear. □

Theorem 25 *The inversion function $t \to t^{-1}$ is operator monotone decreasing on $(0, +\infty)$. (That is, when $0 \leq A \leq B$ in $\mathcal{B}(\mathcal{H})$ and A is invertible, then $A^{-1} \geq B^{-1}$.)*

Proof: Let A and B be self-adjoint with spectra contained in $(0, +\infty)$, and suppose $A \leq B$. Then $0 \leq A \leq B$, and A and B are invertible. If $B = I$, then the result holds by the lemma above. For the general case, observe that

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from the functional calculus we get $B^{-1/2} = (B^{1/2})^{-1} \geq 0$, so that

$$A \leq B \quad \text{implies} \quad B^{-1/2}AB^{-1/2} \leq B^{-1/2}BB^{-1/2} = I.$$

Then from the previous case we have

$$B^{-1/2}AB^{-1/2} \leq I \quad \text{if and only if} \quad B^{1/2}A^{-1}B^{1/2} \geq I,$$

which holds if and only if $A^{-1} \geq B^{-1}$. □

Theorem 26 *The square root function $A \to A^{1/2}$ is operator monotone increasing on* $[0, +\infty)$.

Proof: (After G. K. Pedersen [PGK, p. 102]) Let A and B be self-adjoint with spectra contained in $[0, +\infty)$, and let $A \leq B$. Then A and B are positive.

Suppose first that A and B are invertible. Now for positive invertible S and T we have

$$\begin{aligned}
S \leq T \text{ if and only if } & T^{-1/2}S^{1/2}S^{1/2}T^{-1/2} = T^{-1/2}ST^{-1/2} \leq I \\
\text{if and only if } & (S^{1/2}T^{-1/2})^*(S^{1/2}T^{-1/2}) \leq I \\
\text{if and only if } & \|(S^{1/2}T^{-1/2})^*(S^{1/2}T^{-1/2})\| \leq 1 \\
\text{if and only if } & \|S^{1/2}T^{-1/2}\| \leq 1,
\end{aligned}$$

so that $\|A^{1/2}B^{-1/2}\| \leq 1$ holds. Since $\|SS^*\| = \|S\|^2$ for all S, we then have for all positive integers n that

$$\begin{aligned}
\|A^{1/4}B^{-1/4}\|^{2^{n+1}} &= \|B^{-1/4}A^{1/2}B^{-1/4}\|^{2^n} = \|(B^{-1/4}A^{1/2}B^{-1/4})^{2^n}\| \\
&= \|B^{-1/4}(A^{1/2}B^{-1/2})^{2^n-1}A^{1/2}B^{-1/4}\|
\end{aligned}$$

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$$\leq \|B^{-1/4}\|^2 \|A^{1/2}\| \|A^{1/2}B^{-1/2}\|^{2^n-1} \leq \|B^{-1/4}\|^2 \|A^{1/2}\|,$$

since $\|A^{1/2}B^{-1/2}\| \leq 1$. It follows that for all n we have

$$\|A^{1/4}B^{-1/4}\| \leq \|B^{-1/4}\|^{2/2^{n+1}} \|A^{1/2}\|^{1/2^{n+1}}.$$

Taking the limit on n, we get

$$\|A^{1/4}B^{-1/4}\| \leq 1,$$

which implies that $A^{1/2} \leq B^{1/2}$.

For the non-invertible case, let $\epsilon > 0$, so that $(A+\epsilon I)^{1/2} \leq (B+\epsilon I)^{1/2}$, by the invertible case. Now $(\epsilon I)^{1/2} = \epsilon^{1/2} I \leq (B+\epsilon I)^{1/2}$ (also by the invertible case, or by the functional calculus). Now for any $t > 0$,

$$\sqrt{t+\epsilon} - \sqrt{t} = \frac{\left(\sqrt{t+\epsilon} - \sqrt{t}\right)\left(\sqrt{t+\epsilon} + \sqrt{t}\right)}{\sqrt{t+\epsilon} + \sqrt{t}}$$

$$\frac{\epsilon}{\sqrt{t+\epsilon} + \sqrt{t}} \leq \frac{\epsilon}{\sqrt{\epsilon}} = \sqrt{\epsilon}.$$

From the functional calculus we have for any positive S that

$$0 \leq (S+\epsilon I)^{1/2} - S^{1/2} \leq \epsilon^{1/2} I,$$

and it follows that $\|(S+\epsilon I)^{1/2} - S^{1/2}\| \to 0$ as ϵ goes down to zero. From $(A+\epsilon I)^{1/2} \leq (B+\epsilon I)^{1/2}$ it follows then that $A^{1/2} \leq B^{1/2}$. □

Exercise Show that the function

$$f_\gamma(t) = t(1+\gamma t)^{-1} = \gamma^{-1}(1-(1+\gamma t)^{-1})$$

is operator monotone on $[0, +\infty)$ if $\gamma \geq 0$ and operator monotone on $[0, 1]$ if $-1 < \gamma \leq 0$.

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Remark 11 Using the result of the previous exercise, it can be shown (see for example [PGKII, p. 8]) that if $0 < \gamma \leq 1$, then the function $t \to t^\gamma$ is operator monotone on $[0, +\infty)$.

Remark 12 By the Gelfand-Naimark Theorem, every C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. It follows easily from this that the inverse and square root functions are actually operator monotone on any C^*-algebra with identity.

Exercise

1. Show that the function

$$f_\gamma(t) = t(1 + \gamma t)^{-1} = \gamma^{-1}(1 - (1 + \gamma t)^{-1})$$

is operator monotone on $[0, +\infty)$ if $\gamma \geq 0$ and operator monotone on $[0, 1]$ if $-1 < \gamma \leq 0$.

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Partial Isometries, Positivity and the Polar Decomposition of an Operator

Each non-zero complex number z has a unique decomposition (the "polar decomposition") into the product of a non-negative real number (its magnitude) and a number $e^{i\vartheta}$ of magnitude one. (The number ϑ is not unique, but $e^{i\vartheta}$ is.) We shall see in this section that there is a generalization of this decomposition to an arbitrary bounded linear operator A on a Hilbert space $\mathcal{H}$. A reader of the discussion of positive operators above will probably guess (correctly) that the generalization of the magnitude of z is the unique positive square root of A^*A. The analogue of $e^{i\vartheta}$ might at first glance seem to be a unitary operator, but this turns out to be too restrictive. (Consider for example the unilateral shifts.)

Definition 24 Let U be an element of $\mathcal{B}(\mathcal{H})$. We say that U is a *partial isometry* if U is isometric on the orthogonal complement of its kernel. If U is a partial isometry, then the orthogonal complement of its kernel is its *initial space*, and the range of U is its *final space*.

Note that the final space of a partial isometry must be closed, since a partial isometry U is isometric on its (complete) initial space. The class of all partial isometries includes all orthogonal projections, all unitary operators, and the unilateral and bilateral shifts. There is a further connection between projections and partial isometries; in fact, an operator U is a partial isometry if and only if U^*U is a projection. We prove this in a sequence of propositions and corollaries. Since the class of partial isometries is an important one, we include here a more thorough and systematic treatment than is necessary

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just to obtain the polar decomposition.

Proposition 43 *Let U be an element of $\mathcal{B}(\mathcal{H})$, and suppose U^*U is a projection. Then UU^* is also a projection.*

Proof: Multiplying $U^*UU^*U = U^*U$ on the right by U^* gives

$$U^*UU^*UU^* = U^*UU^*.$$

Thus

$$U^*(UU^*UU^* - UU^*) = 0,$$

so

$$\text{kernel}(U^*) \supseteq R(UU^*UU^* - UU^*).$$

But

$$R(UU^*UU^* - UU^*) \subseteq R(U) \subseteq \text{kernel}(U^*)^\perp.$$

Thus

$$R(UU^*UU^* - UU^*) = \{0\},$$

whence we get

$$UU^*UU^* = UU^*.$$

□

Corollary 30 *Let U be an element of $\mathcal{B}(\mathcal{H})$. Then UU^* is an orthogonal projection if and only if U^*U is an orthogonal projection.*

Proposition 44 *Let U be an element of $\mathcal{B}(\mathcal{H})$. If U^*U is an orthogonal projection, then U is a partial isometry, and U^*U has range equal to the orthogonal complement of the kernel of U.*

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Proof: Let $E = U^*U$ be an orthogonal projection. Then for all x in $\mathcal{H}$,

$$\|Ux\|^2 = \langle Ux, Ux\rangle = \langle U^*Ux, x\rangle = \langle Ex, x\rangle = \langle Ex, Ex\rangle = \|Ex\|^2,$$

so U and E have the same kernel. Let K be the orthogonal complement of this kernel, so that $K = R(E)$. For all x in K we have

$$\|Ux\|^2 = \langle Ux, Ux\rangle = \langle U^*Ux, x\rangle = \langle Ex, x\rangle = \langle x, x\rangle = \|x\|^2,$$

so that U is isometric on the orthogonal complement of its kernel. □

Proposition 45 *Let U be a partial isometry in $\mathcal{B}(\mathcal{H})$. Then U^*U is an orthogonal projection.*

Proof: Let K be the orthogonal complement of the kernel of U, and let E be the orthogonal projection onto K. Then the kernel of E is equal to the kernel of U, and the range of E is the closure of the range of U^*. Moreover, by hypothesis we have for all x in K that

$$\langle U^*Ux, x\rangle = \langle Ux, Ux\rangle = \|Ux\|^2 = \|x\|^2 = \langle x, x\rangle = \langle Ex, x\rangle.$$

If now y is orthogonal to K, then y is an element of the kernel of U, so that all of the numbers $\langle U^*Ux, y\rangle$, $\langle U^*Uy, x\rangle$, and $\langle U^*Uy, y\rangle$ are zero. Similarly, the numbers $\langle Ex, y\rangle$, $\langle Ey, x\rangle$, and $\langle Ey, y\rangle$ are all zero. Thus for x in K and y orthogonal to K we have

$$\langle U^*U(x+y), x+y\rangle = \langle U^*Ux, x\rangle = \langle Ex, x\rangle = \langle E(x+y), x+y\rangle.$$

It follows that $U^*U = E$, so U^*U is an orthogonal projection. □

Corollary 31 *Let U be an element of $\mathcal{B}(\mathcal{H})$. Then U is a partial isometry if and only if U^*U is an orthogonal projection.*

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Corollary 32 *Let U be an element of $\mathcal{B}(\mathcal{H})$. Then U is a partial isometry if and only if U^* is a partial isometry.*

Thus U is a partial isometry if and only if U^* is a partial isometry if and only if U^*U is an orthogonal projection if and only if UU^* is an orthogonal projection.

Suppose now that U is a partial isometry. We saw above that the range of U^*U is the orthogonal complement of the kernel of U. Since U^* is also a partial isometry, the range of U^* is closed, and hence we see that

$$R(U^*U) = \ker(U)^\perp = R(U^*)$$

In particular, the final space of U^* must agree with the initial space of U. By interchanging the roles of U and U^*, we see that we have established the following result.

Theorem 27 *Let U be an element of $\mathcal{B}(\mathcal{H})$. Then U is a partial isometry if and only if U^* is a partial isometry, and U is a partial isometry if and only if U^*U is an orthogonal projection, in which case the ranges of U^* and of U^*U coincide with the initial space of U. If U is a partial isometry, then the final space of U^* agrees with the initial space of U, and the final space of U agrees with the initial space of U^*.*

Thus the ranges of U and U^* are both closed whenever either is a partial isometry. Since U^*U is the projection onto the range of U^* and UU^* is the projection onto the range of U, the restrictions of U and U^* to their initial spaces can be regarded as isometries between the Hilbert spaces $R(U^*)$ and $R(U)$, and these restrictions act as inverse maps for one another between

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these two spaces. It is easy to see (exercise) that given two isometric closed subspaces of $\mathcal{H}$, we get from each isometry between them a partial isometry on $\mathcal{H}$ with these two spaces as its initial and final spaces.

Exercise Let A be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Let R be the relation on the set of all orthogonal projections in A that is defined by the following condition: E is related to F if and only if there is a partial isometry U in A such that $U^*U = E$ and $UU^* = F$. Show that R is an equivalence relation. Show too that if $A = \mathcal{B}(\mathcal{H})$, then E and F are equivalent if and only if there exists a linear isometry from the range of E onto the range of F, i.e., if and only if the ranges of E and F have the same Hilbert space dimension. (This equivalence relation plays an important role in the theory of von Neumann algebras.)

Theorem 28 (the Polar Decomposition) *Let A be an element of $\mathcal{B}(\mathcal{H})$. Then there exists a partial isometry V with initial space $\overline{R(A^*)}$ and final space $\overline{R(A)}$ such that $A = V(A^*A)^{1/2} = (AA^*)^{1/2}V$. If $A = WH$ with H positive and with W a partial isometry whose initial space is $\overline{R(H)}$, then $H = (A^*A)^{1/2}$ and $W = V$. If A and A^* are injective, then V is unitary.*

Proof: For all x in $\mathcal{H}$, we have

$$\langle (A^*A)^{1/2}x, (A^*A)^{1/2}x\rangle = \langle A^*Ax, x\rangle = \langle Ax, Ax\rangle,$$

so the norms of $(A^*A)^{1/2}x$ and Ax agree for every x. In particular, $(A^*A)^{1/2}$ and A have the same kernel, so that there is a well-defined function V_0 given by

$$V_0[(A^*A)^{1/2}x] = Ax.$$

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By the calculation at the beginning of the proof, this function is isometric on the range of $(A^*A)^{1/2}$. It is also linear, since

$$\begin{aligned} V_0[\alpha(A^*A)^{1/2}x_1 + (A^*A)^{1/2}x_2] &= V_0[(A^*A)^{1/2}(\alpha x_1 + x_2)] \\ &= A(\alpha x_1 + x_2) = \alpha A x_1 + A x_2 \\ &= \alpha V_0(A^*A)^{1/2}x_1 + V_0(A^*A)^{1/2}x_2. \end{aligned}$$

It is easy to check (exercise) that V_0 extends in a unique way to an isometric linear map, which we again denote by V_0, of the closure of the range of $(A^*A)^{1/2}$ onto the closure of the range of A. By construction, we have $A = V_0(A^*A)^{1/2}$, and we claim that V_0 has a further extension to a partial isometry V in $\mathcal{B}(\mathcal{H})$. To see this, note that $\overline{R(A^*)} = \ker A^\perp = \ker((A^*A)^{1/2})^\perp = \overline{R((A^*A)^{1/2})}$, let E be the orthogonal projection of $\mathcal{H}$ onto $\ker A^\perp$, and put $Vx = V_0(Ex)$ for each x in $\mathcal{H}$. Then $V = V_0$ on $\ker A^\perp$, so that $A = V(A^*A)^{1/2}$ on all of $\mathcal{H}$, and V is clearly bounded and linear from $\mathcal{H}$ into $\mathcal{H}$. Since V_0 is isometric on the range of E, $Vx = 0$ if and only if $Ex = 0$, so the kernel of V agrees with the kernel of E, i.e., with $\ker(A)$. In particular, V is isometric on the orthogonal complement of its kernel, so V is a partial isometry with initial space $\ker A^\perp = \overline{R((A^*A)^{1/2})}$ and final space $\overline{R(A)}$. Since $\overline{R(A^*)} = \overline{R(A^*A)^{1/2}}$, the existence assertion will be established as soon as we show that $A = (AA^*)^{1/2}V$.

Now $A = V(A^*A)^{1/2}$ implies $A^* = (A^*A)^{1/2}V^*$ and hence $AA^* = VA^*AV^*$. But $(V(A^*A)^{1/2}V^*)^2 = V(A^*A)^{1/2}V^*V(A^*A)^{1/2}V^* = VA^*AV^*$, since V^*V is the projection onto the range of $V^* = \overline{R((A^*A)^{1/2})}$. Thus $(V(A^*A)^{1/2}V^*)^2 = VA^*AV^* = AA^*$. Hence $V(A^*A)^{1/2}V^* = (AA^*)^{1/2}$ by uniqueness of the positive square root of AA^*. This implies that $(AA^*)^{1/2}V = V(A^*A)^{1/2}V^*V =$

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$V(A^*A)^{1/2} = A$, since V^*V is the projection onto the orthogonal complement of the kernel of $(A^*A)^{1/2}$.

If now W and H are as described, then W^*W is the projection onto $\overline{R(H)}$, so $W^*WH = H$, so that $A^*A = HW^*WH = H^2$. Hence $H = (A^*A)^{1/2}$ by uniqueness of square roots. But then W and V have the same initial space and agree on this space, so $W = V$.

If A and A^* both have kernel zero, then the ranges of A^* and A are dense in $\mathcal{H}$. This means that V is a partial isometry with initial space $\mathcal{H}$ and final space $\mathcal{H}$, so V is unitary. □

Remark 13 $(A^*A)^{1/2}$ and $(AA^*)^{1/2}$ both lie in $C^*(A)$, but V need not (exercise). We shall see later that V must always lie in the *von Neumann algebra* generated by A. (See below.) One can show (exercise) that A is normal if and only if $V(A^*A)^{1/2} = (A^*A)^{1/2}V$.

Exercises

1. Let K_1 and K_2 be closed subspaces of a Hilbert space $\mathcal{H}$, and suppose U_0 is an isometric linear map of K_1 onto K_2. Show that there exists a partial isometry U of $\mathcal{H}$ into $\mathcal{H}$ with initial space K_1 and final space K_2 such that $U = U_0$ on K_1.

2. Let A be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, and let P denote the set of all orthogonal projections in A. For E and F in P define $E \sim F$ to mean that there exists a partial isometry in A such that $U^*U = E$ and $UU^* = F$.

   a) Show that $\sim$ is an equivalence relation on P.

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b) Suppose $A = \mathcal{B}(\mathcal{H})$. Show that $E \sim F$ if and only if there exists a linear isometry of the range of E onto the range of F.

3. Let X and Y be Banach spaces, and let X_0 be a subspace of X. Suppose $V_0 : X_0 \to Y$ is isometric and linear. Show that there exists a unique isometric linear map V taking the closure of X_0 onto the closure of $V_0(X_0)$ such that $V = V_0$ on X_0.

4. Let $A \in \mathcal{B}(\mathcal{H})$, and let $A = V(A^*A)^{1/2}$ be its polar decomposition.

   a) Find an example to show that V need not lie in $C^*(A)$.

   b) Show that A is normal if and only if $V(A^*A)^{1/2} = (A^*A)^{1/2}V$.

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The Theorem of Fuglede, Putnam and Rosenblum

The proof of the main result of this section is a nice illustration of the use of the continuous functional calculus, and the result itself will be important for us in a later chapter when we prove our second version (the "measurable functional calculus" version) of the spectral theorem. As we shall see in this chapter, the main result of this section also implies that the C^*-algebra generated by a set of pairwise commuting normal operators is itself commutative.

For any element b of a Banach algebra A with identity I, the ratio test shows that the series $\sum_{k=0}^{+\infty} \frac{b^k}{k!}$ converges absolutely in A. Suppose now that A is a C^*-algebra with identity and that b is normal in A. Since the Maclauren series for e^z converges uniformly to e^z on every compact subset of $\mathbb{C}$, the sum of the series $\sum_{k=0}^{+\infty} \frac{b^k}{k!}$ must be e^b. Note that as the coefficients of this series are real, we have $(e^b)^* = e^{b^*}$. Observe too that since the series $\sum_{k=0}^{+\infty} \frac{z^k b^k}{k!}$ converges to e^{zb} for all complex numbers z, the function $\varphi(e^{zb})$ is analytic on the entire plane for each bounded linear functional φ of A. Since $e^z e^{-z} = 1$ in $C(\text{sp}(b))$, e^{-b} must be an inverse in $C^*(b)$ for e^b.

Exercise Take the series $\sum_{k=0}^{+\infty} \frac{b^k}{k!}$ as the definition of e^b in a Banach algebra with identity, and show that each e^b is invertible with inverse e^{-b}. Use the absolute convergence of the series representations to show that if a and b commute, then $e^{a+b} = e^a e^b$.

Theorem 29 (Fuglede-Putnam-Rosenblum) *Let A be a C^*-algebra with identity I, and let M, N and B be elements of A. Suppose M and N are*

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normal, and suppose that $NB = BM$. Then $N^*B = BM^*$. In particular, if an element B of A commutes with a normal element N of A, then B also commutes with N^*.

Proof: From $NB = BM$, an easy induction argument shows that $N^kB = BM^k$ for all non-negative k. It follows that for any polynomial p, $p(N)B = Bp(M)$.

Now let w be an arbitrary fixed complex number, and let $h(z) = e^{i\bar{w}z}$. Let $\{p_n\}$ be the sequence of Taylor polynomials at zero for h. Since $\{p_n\}$ converges uniformly to h on every compact subset of $\mathbb{C}$, we have $\{p_n\}$ uniformly convergent to h in $C(\mathrm{sp}(M))$ and in $C(\mathrm{sp}(N))$. It follows that

$$\|p_n(M) - e^{i\bar{w}M}\| \to 0 \quad \text{and} \quad \|p_n(N) - e^{i\bar{w}N}\| \to 0$$

in $C^*(M)$ and $C^*(N)$ respectively. In particular, $p_n(M)$ and $p_n(N)$ are norm-convergent in A to $h(M) = e^{i\bar{w}M}$ and $h(N) = e^{i\bar{w}N}$ respectively. As $p(N)B = Bp(M)$ holds for all polynomials p, we must have $e^{i\bar{w}N}B = Be^{i\bar{w}M}$, so that

$$B = e^{-i\bar{w}N}Be^{i\bar{w}M}.$$

Now consider the function

$$f(w) = e^{-iwN^*}Be^{iwM^*}.$$

From the previous paragraph, we have

$$f(w) = e^{-iwN^*}Be^{iwM^*} = e^{-iwN^*}e^{-i\bar{w}N}Be^{i\bar{w}M}e^{iwM^*}.$$

Since N is normal, it follows from remarks above that $e^{-i\bar{w}N}$ and e^{-iwN^*} lie in $C^*(N)$. Recall that the continuous functional calculus is an isometric

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$*$-isomorphism that takes a polynomial in z and $\bar{z}$ to the same polynomial in N and N^*, and takes complex conjugation on $\mathrm{sp}(N)$ to N^*. It follows that it takes the function $e^{-iw\bar{z}} = \sum_{k=0}^{+\infty} \frac{(-iw\bar{z})^k}{k!}$ to $e^{-iwN^*} = \sum_{k=0}^{+\infty} \frac{(-iwN^*)^k}{k!}$ and the function $e^{-i(w\bar{z}+\bar{w}z)} = \sum_{k=0}^{+\infty} \frac{(-i(w\bar{z}+\bar{w}z))^k}{k!}$ to $e^{-i(wN^*+\bar{w}N)}$. Since $e^{-iw\bar{z}}e^{-i\bar{w}z} = e^{-iw\bar{z}-i\bar{w}z} = e^{-i(w\bar{z}+\bar{w}z)}$ for all z in $\mathrm{sp}(N)$, we have

$$e^{-iwN^*}e^{-i\bar{w}N} = e^{-iwN^*-i\bar{w}N} = e^{-i(wN^*+\bar{w}N)}.$$

A similar argument shows that $e^{i\bar{w}M}e^{iwM^*} = e^{i(\bar{w}M+wM^*)}$, and it follows that we have

$$f(w) = e^{-i(wN^*+\bar{w}N)}Be^{i(\bar{w}M+wM^*)}$$

for all complex numbers w. Since $-(wN^* + \bar{w}N)$ and $\bar{w}M + wM^*$ are self-adjoint, $e^{i(\bar{w}M+wM^*)}$ and $e^{-i(wN^*+\bar{w}N)}$ are unitary. Thus for all complex w,

$$\begin{aligned}\|f(w)\| &= \|e^{-i(wN^*+\bar{w}N)}Be^{i(\bar{w}M+wM^*)}\| \\ &\le \|e^{-i(wN^*+\bar{w}N)}\|\|B\|\|e^{i(\bar{w}M+wM^*)}\| \le \|B\|.\end{aligned}$$

In particular, f is bounded.

From the absolute convergence of the power series representations for e^{-iwN^*} and e^{iwM^*} we see that for every bounded linear functional φ, $\varphi \circ f$ is analytic throughout the plane, so by Liouville's Theorem, each $\varphi \circ f$ is constant. Putting w equal to zero shows that $\varphi \circ f(w) = \varphi(B)$ for all w and all φ. It then follows from the Hahn-Banach Theorem that $f(w) = B$ for all w. Thus for all complex w we have

$$B = f(w) = e^{-iwN^*}Be^{iwM^*},$$

so that

$$e^{iwN^*}B = Be^{iwM^*}$$

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for all w in $\mathbb{C}$. If we use the Maclauren series for the exponential function to expand both sides of this equation and then apply an arbitrary bounded linear functional φ to the resulting series, we get

$$\sum_{k=0}^{+\infty} \frac{1}{k!}\,\varphi((iN^*)^k B)w^k = \sum_{k=0}^{+\infty} \frac{1}{k!}\,\varphi(B(iM^*)^k)w^k$$

We may differentiate these series term by term, and setting $w = 0$ in the differentiated series gives

$$\varphi(iN^*B) = \varphi(iBM^*).$$

Once again, from the Hahn-Banach Theorem, we get $iN^*B = iBM^*$, so that $N^*B = BM^*$, and the theorem is proved. □

Corollary 33 *Let A be a C^*-algebra, and let S be a set of pairwise commuting normal elements of A. Then the C^*-algebra generated by S is commutative. In particular, if A has an identity I, then the C^*-algebra generated by S and I is commutative.*

Proof: Let $S\hat{}$ be the union of S and the set of all adjoints of elements of S. Since $ab = ba$ if and only if $b^*a^* = a^*b^*$, it follows from the Fuglede-Putnam-Rosenblum Theorem that the set of all linear combinations of products of members of $S\hat{}$ is a commutative $*$subalgebra of A. Its closure is commutative (by the continuity of multiplication in A) and is clearly the smallest C^*-subalgebra of A which contains S. □

Remark 14 There is a satisfactory theory of Banach algebra-valued analytic functions, which comes with an "analytic functional calculus." In this theory

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much of the core of ordinary (complex-valued) analytic function theory, including Liouville's Theorem and basic results about power series expansions, continues to hold. By making use of this extended theory, we could make substantial simplifications of the proof given above of the Fuglede-Putnam-Rosenblum Theorem, as well the proofs in earlier sections of the non-vacuous nature of the spectrum and the spectral radius formula. As one might suspect from the proofs given here for these results, the Hahn-Banach Theorem gives a way to extend the standard analytic function theory to the more general Banach algebra-valued setting. For an introduction to the more general theory, see for example [RWII].

Theorem 29 was established by Fuglede [FB] with the additional hypothesis that $M = N$. The more general version given here appeared in work of Putnam [PCR]. The proof given here is due to Rosenblum [RM].

Exercises

1. Take the series $\sum_{k=0}^{+\infty} \frac{b^k}{k!}$ as the definition of e^b in a Banach algebra with identity, and show that each e^b is invertible with inverse e^{-b}. Use the absolute convergence of the series representations to show that if a and b commute, then $e^{a+b} = e^a e^b$.

2. Let A be a C^*-algebra with identity, and let $f(w) = e^{-iwN^*} B e^{iwM^*}$, as in the proof of the Fuglede-Putnam-Rosenblum Theorem. Show that for every bounded linear functional ϕ on A, $\phi \circ f$ is analytic throughout the plane.

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3. (Berberian [BSK]) Let $L = \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}$ and $T = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$, acting on $\mathcal{H} \oplus \mathcal{H}$, where N, M and B lie in $\mathcal{B}(\mathcal{H})$ with M and N normal. Deduce Putnam's more general version of Theorem 29 from Fuglede's version, applied to L and T.

4. Find two by two complex matrices M, N and B such that M and N are normal, $NB = BM$ and $NB^* \neq B^*M$. Conclude that the hypotheses of the Fuglede-Putnam-Rosenblum Theorem do not imply that $NB^* = B^*M$ (even though they do imply $N^*B = BM^*$). One can even arrange this (see [RWII]) with M and N self-adjoint and T normal.

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Homomorphisms and representations

In this section we show that a $*$-homomorphism between two C^*-algebras with identity is necessarily norm-decreasing, and is actually isometric whenever it is injective. First we need a few technical results about the spectra of elements in algebras and subalgebras. These results are obvious when the identity of the larger algebra lies in the subalgebra, but require some checking if the subalgebra has a different identity. Along the way we shall obtain a strengthening of spectral permanence for C^*-algebras.

Let B be an algebra with identity I, and let A be a subalgebra of B with identity E. Then $(I-E)^2 = I - 2E + E^2 = I - E$. Moreover, for each a in A we have $a(I-E) = aI - aE = a - a = 0$, and similarly $(I-E)a = 0$. Thus we have for all a_1 and a_2 in A and all μ_1 and μ_2 in $\mathbb{C}$ that

$$[a_1 + \mu_1(I-E)][a_2 + \mu_2(I-E)] = a_1a_2 + \mu_1\mu_2(I-E).$$

Let A_I denote the subalgebra of B generated by A and I (which agrees with the subalgebra of B generated by A and $I-E$).

Proposition 46 *With the hypotheses above, A_I is the algebra direct sum of the subalgebras A and $\mathbb{C}(I-E)$. Moreover, if B is a normed algebra and A is a closed subalgebra of B, then A_I is closed in B.*

Proof: We may assume that I and E are distinct, since $I = E$ implies that $A = A_I$ and $\mathbb{C}(I-E) = \{0\}$.

If $a_1 + \mu_1(I-E) = a_2 + \mu_2(I-E)$ with a_1 and a_2 in A and μ_1 and μ_2 scalars, then $a_1 - a_2 = (\mu_2 - \mu_1)(I-E)$. Multiplying by $I-E$ gives $0 = (\mu_2-\mu_1)(I-E)$, so that $a_1 = a_2$, and then $\mu_1 = \mu_2$. It follows easily

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that A_I is the vector space direct sum of A and $\mathbb{C}(I-E)$, and the remarks above show that the multiplication on A_I agrees with that from the direct sum of A and $\mathbb{C}(I-E)$.

Suppose now that the sequence $\{a_n + \mu_n(I-E)\}$ in A_I converges in B to b. Then multiplying by $I-E$ shows that $\{\mu_n(I-E)\}$ converges to $b(I-E)$. Since $\mathbb{C}(I-E)$ is one dimensional, and hence closed in B, we have $b(I-E) = \mu(I-E)$ for some μ in $\mathbb{C}$. Now $a_n = [a_n + \mu_n(I-E)] - \mu_n(I-E)$ converges in B to $b - b(I-E) = bE$. Since A is closed in B, we have $bE \in A$. But then $b = bE + b(I-E) = bE + \mu(I-E)$ lies in A_I, so A_I is closed in B. □

Note in particular that if B is a C^*-algebra and A is a C^*-subalgebra of B, then A_I is also a C^*-subalgebra of B.

Proposition 47 *Suppose E and I are distinct. Then an element $a+\mu(I-E)$ of A_I is invertible in A_I if and only if $\mu \neq 0$ and a is invertible in A. Moreover, for each a in A, the spectrum of a in A_I is equal to the union of the spectrum of a in A with $\{0\}$.*

Proof: The first assertion follows from the fact that A_I is the direct sum of A and $\mathbb{C}(I-E)$. Since $a - \lambda I = a - \lambda E - \lambda(I-E)$ we see that $a - \lambda I$ is invertible in A_I if and only if $a - \lambda E$ is invertible in A and $\lambda \neq 0$. Thus the resolvent set of a in A_I is equal to the resolvent set of a in A intersected with the complement of $\{0\}$, so the spectrum of a in A_I is equal to $\mathrm{sp}_A(a) \cup \{0\}$. □

Corollary 34 *Let A, B, E and I be as above. If $E = I$, then for each a in A, $\mathrm{sp}_B(a) \subseteq \mathrm{sp}_A(a)$. If E and I are distinct, then for each a in A, $\mathrm{sp}_B(a) \subseteq \mathrm{sp}_A(a) \cup \{0\}$.*

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Proof: Since A_I is a subalgebra of B containing the identity of B, the spectrum of a in B is contained in the spectrum of a in A_I. The result now follows directly from the proposition when $E \neq I$, and from the fact that $A = A_I$ when $I = E$. □

Corollary 35 *Suppose B is a C^*-algebra with identity I, A is a C^*-subalgebra of B with identity E, and $I \neq E$. Then for each a in A, $\mathrm{sp}_B(a) = \mathrm{sp}_A(a) \cup \{0\}$.*

Proof: Since A_I is a C^*-algebra, the spectra of a in A_I and in B coincide, by spectral permanence. The result now follows from the proposition. □

The last corollary is an extension of spectral permanence. Since no element of A can ever be invertible in B unless $I = E$, we may regard it as the strongest extension that we could have hoped to prove. It tells us that for C^*-algebras, the only difference between the spectra of a in A and in B is that which is forced on us by the difference in identities.

Example 4 Let $k < n$, and inject the k by k matrices over $\mathbb{C}$ into the n by n matrices over $\mathbb{C}$ by putting M in the upper left k by k corner of an n by n matrix L and writing zero in every other entry of L. Then the set of eigenvalues of L is the union of $\{0\}$ with the set of eigenvalues of M.

Theorem 30 *Let A be a C^*-algebra with identity E, and let B be a C^*-algebra with identity I. Let π be a $*$-homomorphism of A into B. Then π is norm-decreasing. If π is also injective, then π is isometric.*

Proof: Since $\|\pi(a)\|^2 = \|\pi(a)^*\pi(a)\| = \|\pi(a^*a)\|$ and $\|a\|^2 = \|a^*a\|$ for all a in A, it suffices to check the results on self-adjoint elements a. Thus we need

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only verify that when a is self-adjoint in A, then the spectral radii of a and of $\pi(a)$ satisfy $\rho(\pi(a)) \leq \rho(a)$ in general and satisfy $\rho(\pi(a)) = \rho(a)$ when π is injective.

Now $\pi(A)$ is a subalgebra of B with identity $\pi(E)$. By the results above, $\mathrm{sp}_B(\pi(a)) \subseteq \mathrm{sp}_{\pi(A)}(\pi(a)) \cup \{0\}$. But $a - \lambda E$ invertible in A implies that $\pi(a) - \lambda\pi(E)$ is invertible in $\pi(A)$, so $\mathrm{sp}_{\pi(A)}(\pi(a)) \subseteq \mathrm{sp}_A(a)$. Thus $\mathrm{sp}_B(\pi(a)) \subseteq \mathrm{sp}_A(a) \cup \{0\}$. In particular, $\rho(\pi(a)) \leq \rho(a)$.

Suppose now that π is injective and a is self-adjoint. For each polynomial p, we have $p(\pi(a)) = \pi(p(a))$, where the constant term in $\pi(p(a))$ is a scalar multiple of $\pi(E)$. Thus π maps the algebra of all polynomials in a onto the algebra of all polynomials in $\pi(a)$. Since π is continuous, it must map $C^*(a)$ into the C^*-algebra K generated by $\pi(a)$ and $\pi(E)$, i.e., into the closure K of the polynomials in $\pi(a)$. Since π is an isomorphism of $C^*(a)$ onto its image $\pi(C^*(a))$ in K, we have (by spectral permanence) $\mathrm{sp}_A(a) = \mathrm{sp}_{C^*(a)}(a) = \mathrm{sp}_{\pi(C^*(a))}(\pi(a))$. Since $\pi(C^*(a))$ contains the identity of K, $\mathrm{sp}_K(\pi(a)) \subseteq \mathrm{sp}_{\pi(C^*(a))}(\pi(a))$, so $\mathrm{sp}_K(\pi(a)) \subseteq \mathrm{sp}_A(a)$. If $\mathrm{sp}_K(\pi(a)) \neq \mathrm{sp}_A(a)$, then there exists a non-zero continuous function f on $\mathrm{sp}_A(a)$ that vanishes on $\mathrm{sp}_K(\pi(a))$. We would then have $f((\pi(a)) = 0$ but $f(a) \neq 0$. Let $\{p_n\}$ be a sequence of polynomials that converges uniformly to f on $\mathrm{sp}_A(a)$, so that $p_n(a)$ converges in A to $f(a)$. Since π is continuous, $\pi(p_n(a))$ converges to $\pi(f(a))$. But $\{p_n\}$ also converges uniformly to f on $\mathrm{sp}_K(\pi(a)) \subseteq \mathrm{sp}_A(a)$, so that $\pi(p_n(a)) = p_n(\pi(a))$ converges to zero, which means that $\pi(f(a)) = 0$, contradicting the injectivity of π. Thus $\mathrm{sp}_K(\pi(a)) = \mathrm{sp}_A(a)$. Since $\mathrm{sp}_B(\pi(a)) = \mathrm{sp}_K(\pi(a)) \cup \{0\}$, we have $\mathrm{sp}_B(\pi(a)) = \mathrm{sp}_A(a) \cup \{0\}$, and it follows that $\rho(\pi(a)) = \rho(a)$ as desired. □

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Corollary 36 *If π is an injective $*$-homomorphism of a C^*-algebra A with identity into a C^*-algebra B with identity, then the image of π is closed in B, and hence is a C^*-subalgebra of B.*

Remarks This theorem and corollary are true even if A and B do not possess identities, since one can always adjoin identities to C^*-algebras. (See for example [].) The corollary is also true even if π is not injective. (See [].) The theorem is a particularly nice example of an automatic continuity result, in which the continuity of a class of maps is guaranteed by some other condition (here the C^*-identity). At the root of the argument for the continuity is the fact that the norm in a commutative C^*-algebra is given by the (purely algebraic) spectral radius. Note that the continuity of the evaluation maps $x \to f(x)$ (i.e., the characters) on $C(X)$ is a special case of the theorem just proven.

One special kind of $*$-homomorphism will prove to be particularly important to us.

Definition 25 Let $\mathcal{H}$ be a Hilbert space, and let A be a C^*-algebra. A *representation* (or $*$-*representation*) of A on $\mathcal{H}$ is a $*$-homomorphism of A into $\mathcal{B}(\mathcal{H})$. An injective representation of A is said to be *faithful*. If A has an identity I, then a representation π of A on $\mathcal{H}$ is *non-degenerate* if $\pi(I)$ is the identity operator on $\mathcal{H}$.

By the proposition above, a representation of a C^*-algebra with identity is always norm-decreasing, and is isometric whenever it is faithful. One of the strategies for studying a given C^*-algebra is to study its representations,

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which are more concrete and may yield information more easily than the algebra itself. In the sequel we shall use representations associated with projection-valued measures to obtain a measurable functional calculus.

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The weak and strong operator topologies

Our next goal is to show that every normal operator in $\mathcal{B}(\mathcal{H})$ has a natural representation as an integral with respect to a projection-valued measure. In order to do this we will need to be able to form the sum of an infinite family of orthogonal projections. Now a bounded non-decreasing sequence of real numbers converges in $\mathbf{R}$ to its least upper bound, and an infinite series of non-negative real numbers converges in $\mathbf{R}$ if and only if its sequence of partial sums is bounded above. It would be useful to have similar theorems for sequences and series of operators in $\mathcal{B}(\mathcal{H})$.

Example 5 Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space, and let $\{e_n : n = 1, 2, 3, \ldots\}$ be an orthonormal basis for $\mathcal{H}$. Let E_n be the orthogonal projection onto the span of $\{e_1, e_2, \ldots, e_n\}$. Then the sequence $\{E_n\}$ is non-decreasing, and $E_n \leq I$ for every n. It is easy to see (exercise) that if A is any positive operator in $\mathcal{B}(\mathcal{H})$ that dominates each E_n, then $I \leq A$. Thus I is a least upper bound in $\mathcal{B}(\mathcal{H})$ for the sequence $\{E_n\}$. On the other hand, the norm of $I - E_n$ is always one, so $\{E_n\}$ does not converge in norm to its least upper bound.

Thus we cannot hope that bounded monotone sequences will always be norm convergent in $\mathcal{B}(\mathcal{H})$. In order to get the results that we need we must weaken the topology on $\mathcal{B}(\mathcal{H})$, thereby making it easier for a sequence (or net) to converge.

Definition 26 The *weak operator topology* is the weakest topology (i.e., the topology with the fewest open sets) on $\mathcal{B}(\mathcal{H})$ that makes each of the semi-

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norms

$$T \to |\langle Tx, y\rangle|, \qquad x \in \mathcal{H}, \quad y \in \mathcal{H}$$

continuous. The *strong operator topology* is the weakest topology on $\mathcal{B}(\mathcal{H})$ that makes each of the seminorms

$$T \to \|Tx\|, \qquad x \in \mathcal{H}$$

continuous.

A net $\{T_\gamma\}$ converges weak operator to T in $\mathcal{B}(\mathcal{H})$ if and only if for all x and y in $\mathcal{H}$ we have $|\langle (T_\gamma - T)x, y\rangle| \to 0$, i.e., if and only if for all x and y in $\mathcal{H}$ we have $\langle (T_\gamma - T)x, y\rangle \to 0$, i.e., if and only if for all x and y in $\mathcal{H}$ we have $\langle T_\gamma x, y\rangle \to \langle Tx, y\rangle$. The net $\{T_\gamma\}$ converges strong operator to T in $\mathcal{B}(\mathcal{H})$ if and only if for all x in $\mathcal{H}$ we have $\|(T_\gamma - T)x\| \to 0$, i.e., if and only if for all x in $\mathcal{H}$ we have $\|T_\gamma x - Tx\| \to 0$. Thus the weak operator topology is the topology of pointwise convergence when $\mathcal{H}$ is given its weak topology, and the strong operator topology is the topology of pointwise convergence when $\mathcal{H}$ is given its norm topology.

Exercise Show that in the strong operator topology a neighborhood base at T is given by the family of all sets of the form

$$\{S \in \mathcal{B}(\mathcal{H}) : \|(S - T)x_i\| < \epsilon\}, \qquad i = 1, 2, \ldots, n,$$

where n is a positive integer, $\epsilon > 0$, and $\{x_1, x_2, \ldots, x_n\}$ is a subset of $\mathcal{H}$. Show too that in the weak operator topology a neighborhood base at T is given by

$$\{S \in \mathcal{B}(\mathcal{H}) : |\langle (S - T)x_i, y_i\rangle| < \epsilon\}, \qquad i = 1, 2, \ldots, n,$$

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where n is a positive integer, $\epsilon > 0$, and $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ are subsets of $\mathcal{H}$.

Remark 15 For each x and y in $\mathcal{H}$ and each S and T in $\mathcal{B}(\mathcal{H})$ we have

$$|\langle (S-T)x, y\rangle| \leq \|(S-T)x\|\|y\| \leq \|(S-T)\|\|x\|\|y\|.$$

It follows that norm convergence implies strong operator convergence and that strong operator convergence implies weak operator convergence. (We shall see below that neither of these implications is reversible in general.) This means that the weak operator topology is (strictly) weaker than the strong operator topology and that the strong operator topology is (strictly) weaker than the norm topology.

Since the weak and strong operator topologies on $\mathcal{B}(\mathcal{H})$ do not come from a norm, they do not make $\mathcal{B}(\mathcal{H})$ into a Banach algebra. Nevertheless, the operations in $\mathcal{B}(\mathcal{H})$ do have some continuity properties with respect to these topologies, and these properties are important. The reader should verify all of the assertions below (exercise).

1) The weak and strong operator topologies make the addition and scalar multiplication in $\mathcal{B}(\mathcal{H})$ jointly continuous: if $x_\gamma \to x$ and $y_\gamma \to y$ in H, and if $\mu_\gamma \to \mu$ in $\mathbf{C}$, then $x_\gamma + y_\gamma \to x + y$ and $\mu_\gamma x_\gamma \to \mu x$ in H.

2) The strong operator topology makes the product in $\mathcal{B}(\mathcal{H})$ jointly continuous on bounded subsets; in fact if S_γ is a norm-bounded net and T_γ is any net, and if $S_\gamma \to S$ and $T_\gamma \to T$ strong operator, then $S_\gamma T_\gamma \to ST$ strong operator (so that the product is jointly continuous on $K \times \mathcal{B}(\mathcal{H})$ for every norm-bounded subset K of $\mathcal{B}(\mathcal{H})$).

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3) The strong operator topology makes the product in $\mathcal{B}(\mathcal{H})$ jointly sequentially continuous on all of $\mathcal{B}(\mathcal{H})$: if S_n and T_n are sequences in $\mathcal{B}(\mathcal{H})$ such that $S_n \to S$ and $T_n \to T$ strong operator, then $S_nT_n \to ST$ strong operator. (Use the Principle of Uniform Boundedness.)

4) The weak and strong operator topologies make the product in $\mathcal{B}(\mathcal{H})$ separately continuous: if $S_\gamma \to S$ strong (resp., weak) operator in $\mathcal{B}(\mathcal{H})$, then for every T in $\mathcal{B}(\mathcal{H})$, $S_\gamma T \to ST$ and $TS_\gamma \to TS$ strong (resp., weak) operator.

5) The adjoint operation is weak operator continuous but not strong operator continuous: if $S_\gamma \to S$ weak operator, then $S_\gamma^* \to S^*$ weak operator; however, there exists a sequence $\{S_n\}$ in $\mathcal{B}(\mathcal{H})$ such that $S_n \to 0$ strong operator but S_n^* does not converge strong operator to 0. (Consider the unilateral shifts.)

6) Show that there exists a bounded sequence $\{S_n\}$ in $\mathcal{B}(\mathcal{H})$ such that $\{S_n\}$ and $\{S_n^*\}$ converge weak operator to zero but $\{S_nS_n^*\}$ does not converge weak operator to zero. Conclude that the weak operator topology does not make the product in $\mathcal{B}(\mathcal{H})$ jointly continuous.

7) Let $\{e_n : n = 1, 2, 3, \dots\}$ be a basis for a separable infinite dimensional $\mathcal{H}$. Show that the weak closure of the set $\{\sqrt{n}e_n : n = 1, 2, \dots\}$ contains zero. Thus there exists a subnet $\left\{\sqrt{n_\gamma}e_{n_\gamma}\right\}$ of the sequence $\{\sqrt{n}e_n\}$ that converges weakly in $\mathcal{H}$ to zero. Define a net $\{T_\gamma\}$ in $\mathcal{B}(\mathcal{H})$ by

$$T_\gamma x = \sqrt{n_\gamma}\langle x, e_{n_\gamma}\rangle e_{n_\gamma}$$

Show that $\{T_\gamma\}$ converges strong operator to zero but $\{T_\gamma^2\}$ does not converge strong operator to zero. Conclude that the strong operator topology also fails

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to make the product in $\mathcal{B}(\mathcal{H})$ jointly continuous.

8) Conclude from the results of parts 1) through 7) of this exercise that the weak operator, strong operator, and norm topologies are distinct on every separable infinite dimensional Hilbert space $\mathcal{H}$. Then show that they all coincide on any finite dimensional $\mathcal{H}$.

In the next section, we shall establish the strong operator convergence of norm-bounded monotone nets of positive operators, from which we shall get criteria for the summability of infinite families of positive operators (e.g., projections).

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The convergence of monotone nets

Let $\mathcal{B}(\mathcal{H})_+$ denote the set of all positive operators in $\mathcal{B}(\mathcal{H})$. Observe that $\mathcal{B}(\mathcal{H})_+$ is closed under addition and under multiplication by non-negative real scalars, so that $\mathcal{B}(\mathcal{H})_+$ is a cone in $\mathcal{B}(\mathcal{H})$. If $\{T_\gamma\}$ is a net in $\mathcal{B}(\mathcal{H})_+$ and $\{T_\gamma\}$ converges weak operator to $T \in \mathcal{B}(\mathcal{H})$, then for all x in $\mathcal{H}$, $0 \leq \langle T_\gamma x, x\rangle \to \langle Tx, x\rangle$, so that T is also positive. Thus $\mathcal{B}(\mathcal{H})_+$ is weak operator closed (and hence strong operator closed and norm closed) in $\mathcal{B}(\mathcal{H})$.

Recall that a net $\{T_\gamma\}$ converges weak operator to T in $\mathcal{B}(\mathcal{H})$ if and only if for all x and y in $\mathcal{H}$ we have $\langle T_\gamma x, y\rangle \to \langle Tx, y\rangle$. It follows from polarization that $\{T_\gamma\}$ converges weak operator to T if and only if $\langle T_\gamma x, x\rangle \to \langle Tx, x\rangle$ for all x in $\mathcal{H}$.

Suppose now that $\{T_\gamma\}$ is a norm-bounded net in $\mathcal{B}(\mathcal{H})_+$, and let K be a constant such that $\|T_\gamma\| \leq K$ for all γ. Suppose moreover that $\{T_\gamma\}$ is non-decreasing, i.e., that $\gamma \leq \eta$ implies that $T_\gamma \leq T_\eta$. Then for each x in $\mathcal{H}$, $\langle T_\gamma x, x\rangle$ is non-decreasing and $\langle T_\gamma x, x\rangle \leq K\|x\|^2$ for all γ, so $\langle T_\gamma x, x\rangle$ converges in $\mathbf{R}$. By polarization, $\langle T_\gamma x, y\rangle$ converges in $\mathbf{C}$ for all x and all y. Let $\varphi(x, y) = \lim_\gamma \langle T_\gamma x, y\rangle$. Then φ is a sesquilinear form, and we have

$$|\langle T_\gamma x, y\rangle| \leq \sup_\gamma \|T_\gamma\|\|x\|\|y\| \leq K\|x\|\|y\| \quad \text{for all } x \text{ and } y,$$

which implies

$$|\varphi(x, y)| \leq \sup_\gamma \|T_\gamma\|\|x\|\|y\| \leq K\|x\|\|y\| \quad \text{for all } x \text{ and } y.$$

Thus there exists a unique T in $\mathcal{B}(\mathcal{H})$ such that $\langle Tx, y\rangle = \varphi(x, y)$ for all x and y, and we then have $\{T_\gamma\}$ weak operator convergent to T. (In particular, $T \geq 0$.)

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We claim next that T is a least upper bound for $\{T_\gamma\}$ in $\mathcal{B}(\mathcal{H})_+$. To see this note that as $\langle T_\gamma x, x\rangle$ is non-decreasing and convergent to $\langle Tx, x\rangle$ for every x, T is an upper bound. If also $T_\gamma \leq S$ for all γ, then $\langle T_\gamma x, x\rangle \leq \langle Sx, x\rangle$ for all x and all γ, so that $\langle Tx, x\rangle = \lim_\gamma \langle T_\gamma x, x\rangle \leq \langle Sx, x\rangle$ for all x, so $T \leq S$. Thus T is a least upper bound as claimed. It is easy to see that a least upper bound for $\{T_\gamma\}$ in $\mathcal{B}(\mathcal{H})_+$ is unique if it exists, so we have proved the following result.

Proposition 48 *Let $\{T_\gamma\}$ be a norm-bounded non-decreasing net in $\mathcal{B}(\mathcal{H})_+$. Then $\{T_\gamma\}$ has a least upper bound in $\mathcal{B}(\mathcal{H})_+$, and $\{T_\gamma\}$ converges weak operator to its least upper bound.*

Remark 16 If there exists an S in $\mathcal{B}(\mathcal{H})_+$ such that $T_\gamma \leq S$ for all γ, then $\|T_\gamma\| \leq \|S\|$ for all γ, since $\|T\| = \sup\{\langle Tx, x\rangle : \|x\| \leq 1\}$ whenever T is positive. Conversely, the argument above shows that whenever $\{T_\gamma\}$ is a norm-bounded net in $\mathcal{B}(\mathcal{H})_+$, then $\{T_\gamma\}$ has an upper bound in $\mathcal{B}(\mathcal{H})_+$. Thus a non-decreasing net in $\mathcal{B}(\mathcal{H})_+$ is norm-bounded if and only if it is bounded above in the sense of the ordering in $\mathcal{B}(\mathcal{H})_+$.

We next show that a bounded non-decreasing net in $\mathcal{B}(\mathcal{H})_+$, actually converges strong operator (to its least upper bound).

Proposition 49 *Let $\{T_\gamma\}$ be a net in $\mathcal{B}(\mathcal{H})$. Then $T_\gamma \to 0$ strong operator if and only if $T_\gamma^* T_\gamma \to 0$ weak operator. (Note that we are not assuming the T_γ to be positive, or that the net $\{T_\gamma\}$ is bounded or monotone.)*

Proof: $T_\gamma \to 0$ strong operator if and only if $\|T_\gamma x\| \to 0$ for all x, i.e., if and only if $\|T_\gamma x\|^2 \to 0$ for all x, i.e., if and only if $\langle T_\gamma^* T_\gamma x, x\rangle = \langle T_\gamma x, T_\gamma x\rangle \to 0$

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for all x. But this last is equivalent to $T_\gamma^* T_\gamma \to 0$ weak operator. $\square$

Proposition 50 *Let $\{T_\gamma\}$ be a bounded net in $\mathcal{B}(\mathcal{H})_+$. Then $T_\gamma \to 0$ strong operator if and only if $T_\gamma \to 0$ weak operator if and only if $T_\gamma^{1/2} \to 0$ strong operator. (Here we are not assuming the net to be monotone.)*

Proof: Since $T_\gamma^{1/2}$ is self-adjoint, we have $T_\gamma^{1/2} \to 0$ strong operator if and only if $T_\gamma = (T_\gamma^{1/2})^* T_\gamma^{1/2} \to 0$ weak operator, by the previous proposition. Thus $T_\gamma \to 0$ strong operator implies $T_\gamma \to 0$ weak operator implies $T_\gamma^{1/2} \to 0$ strong operator. Now the net $\{T_\gamma^{1/2}\}$ is also bounded, since $\|T_\gamma^{1/2}\|^2 = \|(T_\gamma^{1/2})^* T_\gamma^{1/2}\| = \|T_\gamma\|$. Since multiplication is jointly strong operator continuous on bounded sets, we see that $T_\gamma^{1/2} \to 0$ strong operator implies that $T_\gamma = T_\gamma^{1/2} T_\gamma^{1/2} \to 0$ strong operator. Thus the three conditions of the proposition are equivalent. $\square$

Corollary 37 *Let $\{T_\gamma\}$ be a bounded net in $\mathcal{B}(\mathcal{H})_+$. Then $T_\gamma \to 0$ weak operator if and only if $T_\gamma^{1/2} \to 0$ weak operator.*

Proof: By the proposition, $T_\gamma \to 0$ weak operator if and only if $T_\gamma^{1/2} \to 0$ strong operator, which implies that $T_\gamma^{1/2} \to 0$ weak operator. If $T_\gamma^{1/2} \to 0$ weak operator, then again by the proposition, $T_\gamma^{1/2} \to 0$ strong operator, so $T_\gamma \to 0$ weak operator. $\square$

Corollary 38 *Let $\{T_\gamma\}$ be a bounded non-decreasing net in $\mathcal{B}(\mathcal{H})_+$. Then $\{T_\gamma\}$ is strong operator convergent (to its weak operator limit).*

Proof: $\{T_\gamma\}$ converges monotonically up in the weak operator topology to its least upper bound T. Then $0 \leq T - T_\gamma$, and $T - T_\gamma$ converges monotonically

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down weak operator to zero. Now $T - T_\gamma$ is bounded above (by T). By the last proposition, we have $(T - T_\gamma)^{1/2} \to 0$ strong operator. Again by this proposition, we have $T - T_\gamma \to 0$ strong operator. But $T - T_\gamma \to 0$ strong operator if and only if $\|(T - T_\gamma)x\| \to 0$ for all x if and only if $T_\gamma \to T$ strong operator. □

Remark 17 Recall from an earlier section that the square root function is *operator monotone* on $(0, +\infty)$, which means that $0 \le S \le T$ implies $S^{1/2} \le T^{1/2}$. Thus $\{T_\gamma^{1/2}\}$ is non-decreasing whenever $\{T_\gamma\}$ is non-decreasing.

It follows easily from the results above that a monotone net in $\mathcal{B}(\mathcal{H})_+$ is strong operator convergent if and only if it is bounded. If now $\{T_\gamma\}_{\gamma\in\Gamma}$ is any family of positive operators, then the collection of all finite subsets F of Γ, ordered by inclusion, is a directed set, and the function $F \to \sum_{\gamma\in F} T_\gamma$ is a non-decreasing net in $\mathcal{B}(\mathcal{H})_+$. In particular, the family $\{T_\gamma\}_{\gamma\in\Gamma}$ is strong operator summable to an element (which we denote by $\sum_{\gamma\in\Gamma} T_\gamma$) of $\mathcal{B}(\mathcal{H})_+$ if and only if the collection of all partial sums of $\sum_{\gamma\in\Gamma} T_\gamma$ is bounded above: in the strong operator topology we have $\sum_{\gamma\in\Gamma} T_\gamma = \lim_F \sum_{\gamma\in F} T_\gamma$.

Exercise Suppose Γ is the disjoint union of Γ_1 and Γ_2. Show that $\sum_{\gamma\in\Gamma} T_\gamma = \sum_{\gamma\in\Gamma_1} T_\gamma + \sum_{\gamma\in\Gamma_2} T_\gamma$.

In particular, let $\{E_\gamma\}_{\gamma\in\Gamma}$ be any family of orthogonal projections in $\mathcal{B}(\mathcal{H})$, and suppose that any two distinct members of the family have orthogonal ranges. Then each finite set of elements of $\{E_\gamma\}_{\gamma\in\Gamma}$ sums to an orthogonal projection (in fact to the orthogonal projection onto the direct sum of the ranges of the summands). Thus the partial sums of $\sum_{\gamma\in\Gamma} E_\gamma$ form

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a collection in $\mathcal{B}(\mathcal{H})_+$ bounded above by the identity I, and hence the family $\{E_\gamma\}_{\gamma\in\Gamma}$ is summable to an element $\sum_{\gamma\in\Gamma} E_\gamma$ of $\mathcal{B}(\mathcal{H})_+$. It is easy to check (exercise) that $\sum_{\gamma\in\Gamma} E_\gamma$ is the orthogonal projection onto the direct sum of the ranges of all the E_γ.

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Monotonicity, continuity and norm convergence

In this section we use a classical result of Dini to establish a useful theorem about the range projection of an operator, and we examine the relationship between norm convergence and monotonicity.

Let A be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, and let A_+ denote the cone of all positive elements of A. By the results of the previous section, every bounded monotone net $\{T_\gamma\}$ in A_+ converges to some positive element T of the strong operator closure of A. If A is strong operator closed in $\mathcal{B}(\mathcal{H})$, or if $\{T_\gamma\}$ converges in norm, then the limit T will certainly lie in A, but in general there is no reason to expect T to be an element of A. This situation is analogous to a familiar one from function theory: if $\{f_\gamma\}$ is a bounded monotone net of real-valued functions in $C(X)$, then $\{f_\gamma\}$ converges pointwise on X to a function f, but the limit f need not be continuous on X.

Example 6 Let $f_n(x) = x^{1/n}$, where $x \in [0,1]$. Then each f_n lies in $C[0,1]$, and $\{f_n\}$ converges pointwise up to the characteristic function of $(0,1]$. Since this characteristic function is not continuous on $[0,1]$, the convergence is not uniform, and the limiting function is not an element of $C[0,1]$.

It turns out that in the presence of compactness and monotonicity, the continuity of the pointwise limit is actually a crucial point. The reader is invited to prove the following theorem. (See for example [].)

Theorem 31 (Dini's Theorem) *Let X be a compact Hausdorff space, and let $\{f_\gamma\}$ be a monotone net of real-valued functions in $C(X)$. Suppose $\{f_\gamma\}$ converges pointwise on X to a continuous function f. Then $\{f_\gamma\}$ converges*

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uniformly on X to f.

Since a uniform limit of continuous functions is always continuous, we have the following result.

Corollary 39 *Let X be a compact Hausdorff space, and let $\{f_\gamma\}$ be a monotone net of real-valued functions in $C(X)$. Suppose $\{f_\gamma\}$ converges pointwise on X to a function f. Then $\{f_\gamma\}$ converges uniformly on X if and only if f is continuous, i.e., if and only if $f \in C(X)$.*

Let us now apply the functional calculus to the sequence $f_n(x) = x^{1/n}$ from the example above.

Theorem 32 *Let T be a positive element of $\mathcal{B}(\mathcal{H})$. Then the sequence $\{T^{1/n}\}$ converges strong operator to the orthogonal projection E of $\mathcal{H}$ onto the closure of the range of T. If T is invertible, or if zero is an isolated point in the spectrum of T, then $\{T^{1/n}\}$ converges to E in norm.*

Proof: Suppose first that $\|T\| \leq 1$, so that $\mathrm{sp}(T) \subseteq [0,1]$. Consider the sequence $\{T^{1/n}\}$, and observe that $T^{1/n} = f_n(T)$, where $\{f_n\}$ is the sequence from the example above. On $[0,1]$, $\{f_n\}$ is non-decreasing and bounded above by the constant function 1, so in $\mathcal{B}(\mathcal{H})_+$ the sequence $\{T^{1/n}\}$ is non-decreasing and bounded above by the identity I. It follows that $\{T^{1/n}\}$ converges strong operator to an element E of $\mathcal{B}(\mathcal{H})_+$. Thus we have $\lim_n T^{2/n} = \lim_n[(T^{1/n})^2] = [\lim_n(T^{1/n})]^2 = E^2$, since multiplication is strong operator jointly continuous on bounded sets. But $T^{1/n} = T^{2/2n}$ for all n, so $\{T^{1/n}\}$ is a subsequence of $\{T^{2/n}\}$, and thus $E^2 = \lim_n T^{2/n} = \lim_n T^{1/n} = E$. It follows that E is an orthogonal projection.

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Now the function $x \to x^{1+1/n}$ is continuous on $[0, 1]$ for every n, and the sequence $\{x^{1+1/n}\}$ converges monotonically up on $[0, 1]$ to the identity function, which is clearly continuous. By Dini's Theorem this convergence is uniform on $[0, 1]$. By the continuous functional calculus, $\{T^{1+1/n}\}$ converges to T in norm (and hence strong operator) in $\mathcal{B}(\mathcal{H})$. But $T^{1+1/n} = (T^{1/n})T$ and $\{T^{1/n}\}$ converges strong operator to E, so we have $T = ET$. It follows immediately that the range of E contains the range of T, and hence contains the closure of the range of T. To show equality of these spaces it therefore suffices to prove that the range of E is contained in the closure of the range of T, or equivalently that the kernel of $E^* = E$ contains the kernel of $T^* = T$. Now kernel$(T) \subseteq$ kernel$(T^{1/2})$, since $Tx = 0$ implies that $0 = \langle Tx, x\rangle = \|T^{1/2}x\|^2$, and hence that $T^{1/2}x = 0$. By induction we get that kernel$(T) \subseteq$ kernel$(T^{1/k})$ whenever k is a power of two. It then follows that whenever $Tx = 0$, then also $Ex = \lim_n T^{1/n}x = 0$, so the kernel of T is contained in the kernel of E. Since E is an orthogonal projection, it is exactly the projection onto the closure of the range of T.

If zero is not an element of the spectrum of T, or if zero is an isolated point of the spectrum of T, then the sequence $\{f_n\}$ converges uniformly on sp(T), so $\{T^{1/n}\}$ converges in norm in $\mathcal{B}(\mathcal{H})$. By the first part of the proof, the limit of $\{T^{1/n}\}$ is the projection E onto the closure of the range of T.

For the general case, we write $T = \lambda S$ with $\lambda > 0$ and $0 \leq S \leq I$. Then by the previous case $\{S^{1/n}\}$ converges to the range projection of S, which coincides with the range projection E of T, and $\{\lambda^{1/n}\}$ converges to 1. It follows that $\{T^{1/n}\}$ converges to E (in whatever sense $\{S^{1/n}\}$ converges). Since $\lambda > 0$, T is invertible if and only if S is invertible, and zero is isolated

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in $\mathrm{sp}(T)$ if and only if zero is isolated in $\mathrm{sp}(S)$.

Recall that if zero is an isolated point of $\mathrm{sp}(T)$, then zero is actually an eigenvalue of T.

The last theorem is significant, since it implies that a strong operator (or weak operator) closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ must contain the range projection of each of its positive elements. (This will also follow from our measurable calculus version of the spectral theorem.) It follows easily (exercise) that such an algebra contains the orthogonal projections onto the kernel and onto the closure of the range of each of its elements. A strong operator closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ that contains the identity operator on $\mathcal{H}$ is called *a von Neumann algebra*. Von Neumann algebras are thus rich in projections. $\square$

Return to the example: Suppose we allow $C[0,1]$ to act on $L^2[0,1]$ by multiplication and put

$$[\pi(g)h](x) = g(x)h(x), \quad \text{where } g \in C[0,1], \quad h \in L^2[0,1], \quad \text{and } x \in [0,1].$$

Then π is a non-degenerate representation of $C[0,1]$ on $L^2[0,1]$, and π is injective, since $C[0,1] \subseteq L^2[0,1]$ and $\pi(g)\bar{g} = |g|^2$. Let A be the image of $C[0,1]$ under π, and let T denote the image under π of the identity function in $C[0,1]$, so that $(Th)(x) = xh(x)$ for all x in $[0,1]$. Then $\mathrm{sp}(T) = [0,1]$, and the range of T is dense in $L^2[0,1]$, since it contains all polynomials with zero constant term. It follows that the range projection of T is the identity. Thus the weak operator limit of the sequence $\{T^{1/n}\}$ actually does lie in the algebra A, even though the corresponding functions in $C[0,1]$ do not converge uniformly.

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Consider however what happens if we replace $[0,1]$ by $[-1,1]$. Let

$$[\pi(g)h](x) = g(x)h(x), \quad \text{where } g \in C[-1,1],\ h \in L^2[-1,1], \text{ and } x \in [-1,1].$$

Let

$$g_n(x) = 0 \text{ for } x \in [-1,0] \quad \text{and} \quad g_n(x) = x^{1/n} \text{ for } x \in [0,1],$$

and let $S = \pi(g_1)$. Then once again, $\{g_n\}$ converges pointwise up (on $[-1,1]$) to the characteristic function of $(0,1]$. Since $g_1 \geq 0$, we have $S \geq 0$, and so $\{S^{1/n}\}$ converges up in $\mathcal{B}(L^2[-1,1])$ to the range projection E of S. (Note that $S^{1/n} = \pi(g_n)$ for every n.) Clearly E is not zero, since $E \geq S \neq 0$. Since $[-1,1]$ is connected, the only non-zero projection in $A = \pi(C[-1,1])$ is the identity. Thus if E were to lie in A, then E would necessarily be the identity operator I on $L^2[-1,1]$. On the other hand, for any function h_0 in $C[-1,1]$ with support in $[-1,0]$ we have $g_n(x)h_0(x) = 0$ for all x in $[-1,1]$, so that $S^{1/n}\pi(h_0) = 0$, so that $E\pi(h_0) = 0$. Since there are evidently many non-zero such h_0, and since π is injective, E cannot be the identity operator. In particular, the C^*-algebra $\pi(C[-1,1])$ is not strong operator closed. By choosing an appropriate sequence in $C[0,1]$ one can show similarly that the algebra $\pi(C[0,1])$ of the preceding paragraph also fails to be strong operator closed in $\mathcal{B}(L^2[0,1])$.

These examples show that when applied to strong operator convergent nets, membership of the limit in the original algebra A is not a reliable indicator of which nets actually converge in norm. It is tempting to look for a generalization of Dini's theorem that can be reliably applied to monotone nets in A_+, and such a theorem does exist. It says roughly that a bounded monotone net $\{T_\gamma\}$ of positive elements in a C^*-algebra A with identity is

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norm convergent if and only if for *every* representation π of A, the strong operator limit of $\{\pi(T_\gamma)\}$ always lies in $\{\pi(A)\}$. This result actually follows from Dini's classical result; the state space of the algebra A plays the role of the compact Hausdorff space X. (See for example [].)

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More on projections

Let A be a $*$-algebra with identity I. Recall that a *projection* (or an *orthogonal projection*) is a self-adjoint idempotent in A, i.e., an element E of A such that $E^2 = E$ and $E^* = E$. By direct calculation we see that $E^2 = E$ if and only if $(I-E)^2 = I - E$, so E is a projection if and only if $I - E$ is a projection. (Recall that I is always self-adjoint, since $Ia^* = a^* = a^*I$ for all a implies that $aI^* = a = I^*a$ for all a, which implies that $I = I^*$ by uniqueness of the identity.) The first proposition below was established earlier, but we repeat it here to make this section self-contained.

Proposition 51 *Let E be a projection in $\mathcal{B}(\mathcal{H})$. Then the range of E coincides with the set of all fixed points of E and with the kernel of $I - E$. In particular, the range of E is closed.*

Proof: Clearly the set of fixed points of E lies in the range of E. If conversely $y = Ex$, then $Ey = E^2x = Ex = y$, so y is fixed by E. Since $(I-E)x = x - Ex$, x is fixed by E if and only if x lies in the kernel of $I - E$. Since $I - E$ is continuous, its kernel is closed. □

Proposition 52 *Let E and F be projections in $\mathcal{B}(\mathcal{H})$. Then E and F have orthogonal ranges if and only if $EF = 0$.*

Proof: The ranges of E and F are orthogonal if and only if we have $\langle Fx, Ey\rangle = 0$ for all x and y in $\mathcal{H}$, i.e., if and only if we have $\langle EFx, y\rangle = 0$ for all x and y in $\mathcal{H}$, i.e. if and only if $EF = 0$. □

Clearly $EF = 0$ if and only if $FE = 0$, so projections with orthogonal ranges commute. Since $E(I-E) = E - E = 0$, E and $I - E$ have orthogonal

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ranges for every projection E in $\mathcal{B}(\mathcal{H})$. It follows that for every projection E in $\mathcal{B}(\mathcal{H})$ and every x in $\mathcal{H}$, we have

$$\|x\|^2 = \|Ex + (I - E)x\|^2 = \|Ex\|^2 + \|(I - E)x\|^2, \quad \text{i.e.,}$$

$$\|x\|^2 = \|Ex\|^2 + \|(I - E)x\|^2.$$

Note: Since $\|E\| = \|E^*E\| = \|E\|^2$, the norm of any non-zero projection E in $\mathcal{B}(\mathcal{H})$ must be one.

Proposition 53 *Let E be a projection in $\mathcal{B}(\mathcal{H})$. Then x lies in the range of E if and only if $\|Ex\| = \|x\|$.*

Proof: Since $\|x\|^2 = \|Ex\|^2 + \|(I - E)x\|^2$, we have

$$\|Ex\| = \|x\| \text{ if and only if } \|(I - E)x\|^2 = 0 \text{ if and only if } (I - E)x = 0.$$

Thus $\|Ex\| = \|x\|$ if and only if x lies in the range of E. □

Theorem 33 *Let E and F be projections in $\mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

1) $E \leq F$
2) $\text{Range}(E) \subseteq \text{Range}(F)$
3) $FE = E$
4) $EF = E$.

Proof: Since the range of the projection F coincides with its set of fixed points, we have $\text{Range}(E) \subseteq \text{Range}(F)$ if and only if $FEx = Ex$ for all x, i.e. if and only if $FE = E$. Taking adjoints shows that $FE = E$ if and only if $EF = E$. Thus 2), 3) and 4) are all equivalent.

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Suppose now that 1) holds. Then $\langle Ex, Ex\rangle = \langle Ex, x\rangle \leq \langle Fx, x\rangle = \langle Fx, Fx\rangle$ for all x, so that $\|Ex\| \leq \|Fx\|$ for all x. In particular, if x lies in the range of E, then

$$\|x\| = \|Ex\| \leq \|Fx\| \leq \|x\|,$$

so that $\|x\| = \|Fx\|$, which means that x lies in the range of F. Thus 1) implies 2).

Finally, suppose 2) holds, so that $\text{Range}(E) \subseteq \text{Range}(F)$. Then $EF = E$, since 2) is equivalent to 4). For all x in $\mathcal{H}$ we have $\|x\|^2 = \|Ex\|^2 + \|(I-E)x\|^2$. Replacing x by Fx we get

$$\|Fx\|^2 = \|EFx\|^2 + \|(I-E)Fx\|^2 \quad \text{for all } x \text{ in } \mathcal{H},$$

so that (since $E = EF$)

$$\|Fx\|^2 = \|Ex\|^2 + \|(I-E)Fx\|^2 \geq \|Ex\|^2 \quad \text{for all } x \text{ in } \mathcal{H}.$$

But this implies that $\langle Ex, x\rangle \leq \langle Fx, x\rangle$ for all x in $\mathcal{H}$, and thus 2) implies 1). □

Corollary 40 *Let E and F be projections in $\mathcal{B}(\mathcal{H})$. If $E \leq F$, then $F - E$ is a projection.*

Proof: Since $E \leq F$, $EF = E = FE$, so $(F-E)^2 = F - FE - EF + E = F - E$. □

Corollary 41 *Let E and F be projections in $\mathcal{B}(\mathcal{H})$. Then $E + F$ is a projection if and only if E and F have orthogonal ranges.*

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Proof: If E and F have orthogonal ranges, then $EF = 0 = FE$, so that $(E+F)^2 = E + EF + FE + F = E + F$, and $E + F$ is a projection.

Conversely, suppose $E + F$ is a projection. Then

$$E + F = (E+F)^2 = E + EF + FE + F,$$

so that $EF + FE = 0$. Thus $EF = -FE$, so that $EF = -EFE$ is self-adjoint. But then $EF = FE$, so that $2EF = 0$, i.e., $EF = 0$. But $EF = 0$ if and only if the ranges of E and F are orthogonal. $\Box$

Theorem 34 *Let* $\{E_\gamma : \gamma \in \Gamma\}$ *be a family of projections in* $\mathcal{B}(\mathcal{H})$*. Then* $\{E_\gamma\}$ *sums to a projection if and only if the* $\{E_\gamma\}$ *have pairwise orthogonal ranges.*

Proof: Suppose the E_γ have pairwise orthogonal ranges. Then the sum S_F over any finite subset F of Γ is a projection, by an induction argument and the last corollary. It follows that the net $\{S_F\}$ of finite sums is non-decreasing and bounded above, and so converges strong operator to its least upper bound E. Since $S_F^2 = S_F$ it follows from the strong operator joint continuity of multiplication on bounded sets that $E^2 = E$, so E is a projection, and thus $\{E_\gamma\}$ sums to a projection.

Conversely, suppose the family $\{E_\gamma\}$ sums to a projection F in $\mathcal{B}(\mathcal{H})$. Let μ and ν be distinct elements of Γ, and let $G = \sum_{\gamma \neq \mu} E_\gamma$. Then $F = E_\mu + G$. By the next to last corollary, G is a projection. By the last corollary, E_μ and G have orthogonal ranges. Now $E_\nu \leq G$, so $\text{Range}(E_\nu) \subseteq \text{Range}(G)$, so the range of E_ν is orthogonal to the range of E_μ. $\Box$

Remark 18 Suppose $E = \sum_\gamma E_\gamma$ is the sum of a pairwise orthogonal family

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$\{E_\gamma\}$ of projections. Then for each x in $\mathcal{H}$, $Ex = \sum_\gamma E_\gamma x$. Since the $E_\gamma x$ are orthogonal, we have for all x in $\mathcal{H}$ that $\|Ex\|^2 = \sum_\gamma \|E_\gamma x\|^2$. Thus when x lies in the range of E, we have $\|x\|^2 = \sum_\gamma \|E_\gamma x\|^2$. Conversely, suppose that $\|x\|^2 = \sum_\gamma \|E_\gamma x\|^2$. Then $\|x\|^2 = \sum_\gamma \|E_\gamma x\|^2 = \|Ex\|^2$, so that $\|x\| = \|Ex\|$, i.e., x lies in the range of E. That is, whenever $\{E_\gamma\}$ is a pairwise orthogonal family, we have $\text{Range}\left(\sum_\gamma E_\gamma\right) = \left\{x \in \mathcal{H} : \|x\|^2 = \sum_\gamma \|E_\gamma x\|^2\right\}$.

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Projection-valued measures and operator-valued integrals

Let T be a normal operator in $\mathcal{B}(\mathcal{H})$. We have seen that the continuous functional calculus is an isometric *-isomorphism of $C(\text{sp}(T))$ into $\mathcal{B}(\mathcal{H})$. We may therefore say, in a slightly different but equivalent language, that the map $f \to f(T)$ is a representation of the C^*-algebra $C(\text{sp}(T))$ on $\mathcal{H}$. Since this map takes the identity of $C(\text{sp}(T))$ to the identity of $\mathcal{B}(\mathcal{H})$, the continuous functional calculus is a non-degenerate representation of $C(\text{sp}(T))$.

Our next goal is to extend this representation to the algebra of all bounded Borel measurable functions on $\text{sp}(T)$. We will see that for a bounded Borel function f, the corresponding operator $f(T)$ (which will have all of the algebraic properties of the function f) will be represented as an integral of f with respect to a certain measure, although the measure in question takes its values in the set of projections on $\mathcal{H}$. We will also see that the resulting measurable functional calculus generalizes the familiar representation of a normal matrix as a linear combination $T = \sum_{\lambda \in \text{sp}(T)} \lambda E_\lambda$ of its eigenspaces.

We begin with the definition of the necessary measure.

Definition 27 Let X be a set, let Ω be a σ-algebra of sets in X, and let $\mathcal{H}$ be a Hilbert space. A *projection-valued measure* (or *resolution of the identity*) for $(X, \Omega, \mathcal{H})$ is a function $E : \Omega \to \mathcal{B}(\mathcal{H})$ such that:

1) for all $\omega \in \Omega$, $E(\omega)$ is a (self-adjoint) projection in $\mathcal{B}(\mathcal{H})$;
2) $E(X) = I$ and $E(\phi) = 0$ (where ϕ denotes the empty set);
3) for all ω_1 and ω_2 in Ω, $E(\omega_1 \cap \omega_2) = E(\omega_1)E(\omega_2)$; and
4) for every sequence $\{\omega_i\}$ of disjoint sets in Ω, $E\left(\bigcup_{i=1}^{+\infty} \omega_i\right) = \sum_{i=1}^{+\infty} E(\omega_i)$.

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Example 7 Let $X = [0, 1]$, let Ω be the σ-algebra of all Borel subsets of X, let $\mathcal{H} = L^2[0, 1]$, and let $E(\omega)$ be the operator on $L^2[0, 1]$ given by multiplication by the characteristic function χ_ω of ω. Then E is a projection-valued measure for $(X, \Omega, \mathcal{H})$.

Remarks Since intersection is a commutative operation on Ω, condition 3) implies that the range of E is a family of commuting projections. Condition 3) also implies that if ω_1 and ω_2 are disjoint, then $E(\omega_1)$ and $E(\omega_2)$ have orthogonal ranges.

A projection-valued measure for $(X, \Omega, \mathcal{H})$ gives rise in a natural way to a large and coherent family of complex measures on the measurable space (X, Ω).

Theorem 35 *If E is a projection-valued measure for $(X, \Omega, \mathcal{H})$, then for each ordered pair (x, y) in $\mathcal{H}$,*

$$E_{x,y}(\omega) = \langle E(\omega)x, y\rangle$$

defines a (countably additive) complex measure on (X, Ω). Each $E_{x,y}$ satisfies

$$\|E_{x,y}\| \leq \|x\|\|y\|.$$

Moreover, each $E_{x,x}$ is a positive measure, and $\|E_{x,x}\| = E_{x,x}(X) = \|x\|^2$.

Proof: It is a straightforward exercise to show that each $E_{x,y}$ is a complex measure.

To bound the total variation $\|E_{x,y}\|$ of $E_{x,y}$ we consider pairwise disjoint measurable subsets $\omega_1, \ldots, \omega_n$ of X. Let $P = \sum_{i=1}^n E(\omega_j)$. Since the ω_j are

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disjoint, the $E(\omega_j)$ are pairwise orthogonal, so P is a projection. For each j, choose θ_j such that $e^{i\theta_j}E_{x,y}(\omega_j) = |E_{x,y}(\omega_j)|$. Then

$$\sum_{j=1}^{n} |E_{x,y}(\omega_j)| = \sum_{j=1}^{n} e^{i\theta_j} E_{x,y}(\omega_j) = \sum_{j=1}^{n} e^{i\theta_j} \langle E(\omega_j)x, y\rangle = \left\langle \sum_{j=1}^{n} E(\omega_j)e^{i\theta_j}x, y \right\rangle.$$

Because the $E(\omega_j)$ are pairwise orthogonal,

$$\left\|\sum_{j=1}^{n} E(\omega_j)e^{i\theta_j}x\right\|^2 = \sum_{j=1}^{n} \|E(\omega_j)e^{i\theta_j}x\|^2 = \sum_{j=1}^{n} \|E(\omega_j)x\|^2 = \|Px\|^2 \le \|x\|^2.$$

Thus

$$\sum_{j=1}^{n} |E_{x,y}(\omega_j)| = \left\langle \sum_{j=1}^{n} E(\omega_j)e^{i\theta_j}x, y \right\rangle \le \left\|\sum_{j=1}^{n} E(\omega_j)e^{i\theta_j}x\right\| \|y\| \le \|x\|\|y\|.$$

Taking the supremum over all such collections $\omega_1, \ldots, \omega_n$ shows that $\|E_{x,y}\| \le \|x\|\|y\|$.

Each $E_{x,x}$ is clearly a positive measure, since

$$E_{x,x}(\omega) = \langle E(\omega)x, x\rangle = \langle E(\omega)x, E(\omega)x\rangle = \|E(\omega)x\|^2.$$

Putting $\omega = X$ gives $\|E_{x,x}\| = E_{x,x}(X) = \langle E(X)x, x\rangle = \langle Ix, x\rangle = \|x\|^2$. □

There seems to be more than enough structure here to define an integral.

Theorem 36 *Let E be a projection-valued measure for $(X, \Omega, \mathcal{H})$, and let f be a bounded measurable function on X. Then there exists a unique operator T in $\mathcal{B}(\mathcal{H})$ such that for all x and y in $\mathcal{H}$,*

$$\langle Tx, y\rangle = \int_X f\, dE_{x,y}.$$

Suppose moreover that $\epsilon > 0$ and that $\{\omega_1, \ldots, \omega_n\}$ is a measurable partition of X such that whenever t and t' lie in the same ω_j, then also $|f(t) - f(t')| \le \epsilon$.

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Then whenever $t_j \in \omega_j$ *for all* j *we have*

$$\Big\| T - \sum_{j=1}^{n} f(t_j)E(\omega_j) \Big\| \le \epsilon.$$

Proof: For each ordered pair (x, y) from $\mathcal{H}$, define $\psi(x, y) = \int_X f\, dE_{x,y}$. Using the fact that each $E_{x,y}$ is a complex measure, it is routine to verify that ψ is a sesquilinear functional on $\mathcal{H}$, and since

$$|\psi(x, y)| = \Big|\int_X f\, dE_{x,y}\Big| \le \|f\|_\infty \|E_{x,y}\| \le \|f\|_\infty \|x\|\|y\|,$$

ψ is bounded. Thus there is a unique operator T in $\mathcal{B}(\mathcal{H})$ such that $\psi(x, y) = \langle Tx, y\rangle$ for all x and y in $\mathcal{H}$. Thus there is a unique operator T in $\mathcal{B}(\mathcal{H})$ such that $\langle Tx, y\rangle = \int_X f\, dE_{x,y}$ for all x and y in $\mathcal{H}$.

To establish the second assertion of the theorem, let $\epsilon > 0$ and consider a measurable partition $\{\omega_1, \ldots, \omega_n\}$ with the given property. Suppose $t_j \in \omega_j$ for all $j = 1, 2, \ldots, n$. Then

$$\begin{aligned}
\Big|\Big\langle \Big(T - \sum f(t_j)E(\omega_j)\Big) x, y \Big\rangle\Big| &= \Big|\langle Tx, y\rangle - \Big\langle \Big(\sum f(t_j)E(\omega_j)\Big) x, y\Big\rangle\Big| \\
&= \Big|\int_X f(t) dE_{x,y}(t) - \sum f(t_j)E_{x,y}(\omega_j)\Big| \\
&= \Big|\sum \int_{\omega_j} f(t) dE_{x,y}(t) - \sum f(t_j)E_{x,y}(\omega_j)\Big| \\
&= \Big|\sum \int_{\omega_j} f(t) dE_{x,y}(t) - \sum \int_{\omega_j} f(t_j) dE_{x,y}(t)\Big| \\
&= \Big|\sum \int_{\omega_j} [f(t) - f(t_j)] dE_{x,y}(t)\Big| \\
&\le \sum \Big|\int_{\omega_j} f(t) - f(t_j) dE_{x,y}(t)\Big| \\
&\le \sum \int_{\omega_j} |f(t) - f(t_j)| d|E_{x,y}|(t)
\end{aligned}$$

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$$\le \sum \epsilon |E_{x,y}|(\omega_j) = \epsilon |E_{x,y}|(X)$$
$$= \epsilon \|E_{x,y}\| \le \epsilon \|x\| \|y\|,$$

since $|f(t) - f(t_j)| \le \epsilon$ whenever $t \in \omega_j$, and since $E_{x,y}$ and $|E_{x,y}|$ are measures. Taking the supremum as x and y range over the unit ball gives $\left\| T - \sum_{j=1}^{n} f(t_j)E(\omega_j) \right\| \le \epsilon$ as required. □

Definition 28 We denote the operator T of the last theorem by $\int_X f dE$, or by $\int_X f(\lambda) dE_\lambda$, and call it the *integral of f with respect to E*. Note that $\int_X f dE = \int_X g dE$ if and only if $\left\langle \left(\int_X f dE \right) x, x \right\rangle = \left\langle \left(\int_X g dE \right) x, x \right\rangle$ for every x in $\mathcal{H}$, i.e., if and only if $\int_X f dE_{x,x} = \int_X g dE_{x,x}$ for every x in $\mathcal{H}$.

Of course, the expression $\sum f(t_j)E(\omega_j)$ is the integral with respect to E of the simple function $\sum f(t_j)\chi_{\omega_j}$. Thus the second assertion of the theorem says that if we uniformly approximate a bounded measurable f by measurable simple functions, then the integral of f with respect to the measure E is the norm limit of the integrals of the approximants. The corresponding theorem for scalar measures is one of the cornerstones of Lebesgue's theory of the integral.

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Projection-valued measures and representations

Let (X, Ω) be a measurable space, and let $B(X)$ denote the space of all bounded Ω-measurable functions from X to the complex numbers. We give $B(X)$ an addition and scalar multiplication defined pointwise, and we employ the supremum norm $\| \ \|_\infty$, defined by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

It is easy to see that $B(X)$ is a Banach space. Since pointwise products and complex conjugates of measurable functions are measurable, $B(X)$ is a C^*-algebra. Observe that if X is a compact Hausdorff space and Ω is the σ-algebra of Borel functions on X, then $C(X)$ is a C^*-subalgebra of $B(X)$.

Now let $\mathcal{H}$ be any Hilbert space. In this section we shall first establish that every projection-valued measure for $(X, \Omega, \mathcal{H})$ gives rise to a representation of $B(X)$ on $\mathcal{H}$. Then we shall show that each representation of $C(X)$ on $\mathcal{H}$ corresponds to a projection-valued measure for $(X, \Omega, \mathcal{H})$, where Ω is the σ-algebra of all Borel subsets of X. Since the continuous functional calculus for any normal operator T on $\mathcal{H}$ is a representation of $C(\mathrm{sp}(T))$ on $\mathcal{H}$, we will then have an extension of the functional calculus to the algebra of all bounded Borel functions on $\mathrm{sp}(T)$. From this we obtain the spectral theorem with a measurable functional calculus.

Theorem 37 *Let E be a projection-valued measure for $(X, \Omega, \mathcal{H})$, and let $\rho : B(X) \to \mathcal{B}(\mathcal{H})$ be defined by $\rho(f) = \int_X f dE$. Then ρ is a non-degenerate representation of $B(X)$. For any x and y in $\mathcal{H}$ we have $\langle \rho(f)x, y \rangle = \int_X f dE_{x,y}$,*

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and $\|\rho(f)x\|^2 = \int_X |f|^2 dE_{x,x}$. *Furthermore, an operator* S *in* $\mathcal{B}(\mathcal{H})$ *commutes with* $\rho(f)$ *for all* f *in* $B(X)$ *if and only if* S *commutes with* $E(\omega)$ *for all* ω *in* Ω.

Proof: The equality $\langle \rho(f)x, y\rangle = \int_X f dE_{x,y}$ follows directly from the definition of $\int_X f dE$. To see that ρ is linear, let f and g be elements of $B(X)$, and let λ be a scalar. For any x in $\mathcal{H}$ we have $\langle \rho(\lambda f + g)x, x\rangle = \langle \big(\int_X (\lambda f + g) dE) x, x\big)\rangle = \int_X (\lambda f + g) dE_{x,x}$. Since each $E_{x,x}$ is a measure, we have $\int_X (\lambda f + g) dE_{x,x} = \lambda \int_X f dE_{x,x} + \int_X g\, dE_{x,x}$ for every x. But this means that for every x we have

$$\begin{aligned}\langle \rho(\lambda f + g)x, x\rangle &= \lambda \left\langle \left(\int_X f dE\right) x, x \right\rangle + \left\langle \left(\int_X g\, dE\right) x, x \right\rangle \\ &= \lambda \langle \rho(f)x, x\rangle + \langle \rho(g)x, x\rangle \\ &= \langle (\lambda\rho(f) + \rho(g))x, x\rangle.\end{aligned}$$

Since x is arbitrary in $\mathcal{H}$, we have $\rho(\lambda f + g) = \lambda\rho(f) + \rho(g)$. Thus ρ is linear.

To see that ρ preserves the adjoint, let f be any element of $B(X)$. Then for any x in $\mathcal{H}$ we have $\langle \rho(\bar{f})x, x\rangle = \int_X \bar{f} dE_{x,x} = \overline{\int_X f dE_{x,x}}$, since $E_{x,x}$ is a (non-negative) measure. Thus $\langle \rho(\bar{f})x, x\rangle = \overline{\int_X f dE_{x,x}} = \overline{\langle \rho(f)x, x\rangle} = \langle x, \rho(f)x\rangle = \langle [\rho(f)]^* x, x\rangle$ for all x, whence we get $\rho(\bar{f}) = [\rho(f)]^*$.

To show that ρ is a representation, it remains only to show that it is multiplicative, so let f and g be elements of $B(X)$, and we shall show that $\rho(fg) = \rho(f)\rho(g)$. Let $\epsilon > 0$, and let $\{\omega_1, \ldots, \omega_n\}$ be a partition of X such that for $h = f$, $h = g$, and $h = fg$ we have $|h(s) - h(t)| \leq \epsilon$ whenever s and t belong to the same set ω_i. Then for any choice of points t_i in ω_i, we have $\| \int_X h dE - \sum_{i=1}^n h(t_i)E(\omega_i)\| \leq \epsilon$ for all three choices of h. It follows that for any such t_i,

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$$\begin{aligned}
\Big\| \int_X fg\,dE - \int_X f dE \int_X g\,dE \Big\| &\leq \Big\| \int_X fgdE - \sum_{i=1}^n f(t_i)g(t_i)E(\omega_i) \Big\| \\
&\quad + \Big\| \sum_{i=1}^n f(t_i)g(t_i)E(\omega_i) - \int_X fdE \int_X gdE \Big\| \\
&\leq \epsilon + \Big\| \sum_{i=1}^n f(t_i)g(t_i)E(\omega_i) - \left(\sum_{i=1}^n f(t_i)E(\omega_i)\right)\left(\sum_{i=1}^n g(t_i)E(\omega_i)\right) \Big\| \\
&\quad + \Big\| \left(\sum_{i=1}^n f(t_i)E(\omega_i)\right)\left(\sum_{i=1}^n g(t_i)E(\omega_i)\right) - \int_X fdE \int_X gdE \Big\|
\end{aligned}$$

Since the $E(\omega_i)$ are pairwise orthogonal projections, the middle term in this last sum is zero. Thus

$$\begin{aligned}
&\Big\| \int_X fgdE - \int_X fdE \int_X gdE \Big\| \\
&\quad \leq \epsilon + \Big\| \left(\sum_{i=1}^n f(t_i)E(\omega_i)\right)\left(\sum_{i=1}^n g(t_i)E(\omega_i)\right) - \int_X fdE \int_X gdE \Big\| \\
&\quad \leq \epsilon + \Big\| \left(\sum_{i=1}^n f(t_i)E(\omega_i)\right)\left(\sum_{i=1}^n g(t_i)E(\omega_i)\right) - \left(\sum_{i=1}^n f(t_i)E(\omega_i)\right) \int_X gdE \Big\| \\
&\qquad + \Big\| \left(\sum_{i=1}^n f(t_i)E(\omega_i)\right) \int_X gdE - \int_X fdE \int_X gdE \Big\| \\
&\quad \leq \epsilon + \Big\| \sum_{i=1}^n f(t_i)E(\omega_i) \Big\| \Big\| \sum_{i=1}^n g(t_i)E(\omega_i) - \int_X gdE \Big\| \\
&\qquad + \Big\| \sum_{i=1}^n f(t_i)E(\omega_i) - \int_X fdE \Big\| \Big\| \int_X gdE \Big\| \\
&\quad \leq \epsilon + \|f\|_\infty \epsilon + \epsilon \|g\|_\infty = \epsilon(1 + \|f\|_\infty + \|g\|_\infty).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\rho(fg) = \rho(f)\rho(g)$ as desired, so ρ is a representation.

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Now $\langle \rho(1)x, x\rangle = \langle \left(\int_X 1\, dE\right) x, x\rangle = \int_X 1\, dE_{x,x} = E_{x,x}(X) = \langle E(X)x, x\rangle = \langle Ix, x\rangle$ for all x in $\mathcal{H}$, so $\rho(1) = I$, and ρ is non-degenerate. Also, since ρ is a representation,

$$\langle \rho(f)x, \rho(f)x\rangle = \langle \rho(f)^*\rho(f)x, x\rangle = \langle \rho(\bar{f}f)x, x\rangle = \langle \rho(|f|^2)x, x\rangle = \int_X |f|^2 dE_{x,x}.$$

As for the commutativity assertion, if S commutes with every $\rho(f)$, then certainly S commutes with every $\rho(\chi_\omega) = E(\omega)$, and if S commutes with every $E(\omega)$, then approximation by simple functions shows that S commutes with every $\rho(f)$. This completes the proof. □

Note that as ρ is a representation and $B(X)$ is a commutative $*$-algebra, the image of ρ consists of normal operators.

Now let X be a compact Hausdorff space. To show that every representation of $C(X)$ gives rise to a projection-valued measure on the Borel subsets of X, we will need a few additional results about complex measures and duals of Banach spaces, all of which can be found in, for example, [CFA].

Observe that if μ is a complex measure on the Borel subsets of X, then the linear functional

$$f \to \int_X f\, d\mu$$

on $C(X)$ is bounded, since $|\int_X f\, d\mu| \leq \|f\|_\infty \|\mu\|$ for every continuous f. Thus each complex measure gives rise in a natural way to an element of the dual space $C(X)^*$ of $C(X)$. In fact every element of this dual space arises in this way:

Theorem 38 (Riesz Representation Theorem for $C(X)^*$) *Let X be a compact Hausdorff space, and let φ be a bounded linear functional on $C(X)$.*

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Then there exists a unique regular complex Borel measure μ on X such that for all f in $C(X)$ we have

$$\varphi(f) = \int_X f \, d\mu.$$

Moreover, the function $\varphi \to \mu$ is an isometric map of $C(X)^*$ onto the space $M(X)$ of all regular complex Borel measures on X, equipped with the total variation norm.

The measures in $M(X)$ are often referred to as the *Radon measures* on X.

Now for each bounded Borel function f on X, the functional

$$f \to \int_X f \, d\mu$$

on $M(X)$ is bounded and linear (this time because $|\int_X f \, d\mu| \leq \|f\|_\infty \|\mu\|$ for every μ in $M(X)$). Thus each (Borel measurable) element f of $B(X)$ gives rise to an element $\hat{f}$ of the dual space of $M(X)$, i.e., to an element of the second dual $C(X)^{**}$ of $C(X)$. If $\{f_\gamma\}$ is a net in $B(X)$ and f is an element of $B(X)$, then clearly $\hat{f}_\gamma \to \hat{f}$ weak* if and only if

$$\int_X f_\gamma d\mu \to \int_X f \, d\mu$$

for every measure μ in $M(X)$.

Now for any bounded Borel measurable function g and any Radon measure μ, the linear functional

$$f \to \int_X fg \, d\mu$$

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on $C(X)$ is also bounded, since $|\int_X fg\,d\mu| \leq \|fg\|_\infty\|\mu\| \leq \|f\|_\infty\|g\|_\infty\|\mu\|$. Thus by the Riesz Theorem there exists a unique measure $g\mu$ in $M(X)$ such that

$$\int_X fg\,d\mu = \int_X f d(g\mu)$$

for all f in $C(X)$. Observe that if $\{f_\gamma\}$ is a net in $B(X)$ and $f \in B(X)$, then $\hat{f}_\gamma \to \hat{f}$ weak* implies that

$$\int_X f_\gamma g\,d\mu = \int_X f_\gamma d(g\mu) \to \int_X f d(g\mu) = \int_X fg\,d\mu$$

for all g in $B(X)$. Since the constant function 1 lies in $B(X)$, it follows at once that $\hat{f}_\gamma \to \hat{f}$ weak* if and only if $\int_X f_\gamma g\,d\mu \to \int_X fg\,d\mu$ for all g in $B(X)$.

Note too that $\hat{f}_\gamma \to \hat{f}$ weak* if and only if $\hat{\bar{f}}_\gamma \to \hat{\bar{f}}$ weak*: if $\hat{f}_\gamma \to \hat{f}$ weak*, then $\int_X \bar{f}_\gamma d\mu = \overline{\int_X f_\gamma d\mu} \to \overline{\int_X f d\mu} = \int_X \bar{f}\,d\mu$ for all non-negative μ, and $\int_X \bar{f}_\gamma d\mu \to \int_X \bar{f} d\mu$ for all μ if and only if $\int_X \bar{f}_\gamma d\mu \to \int_X \bar{f} d\mu$ for all non-negative μ.

It is a consequence of the Hahn-Banach Theorem (see [CFA]) that for any Banach space B, the image of B under the canonical embedding is always weak* dense in B^{**}. (In fact, the image of the norm-closed unit ball in B is dense in the norm closed unit ball of B^{**}; this is Goldstine's Theorem: see for example [CFA, Chapter V, Section 4].) In particular, let us take $B = C(X)$. Then for every bounded Borel function f on X there exists a net $\{f_\gamma\}$ of continuous functions on X such that for all Radon measures μ we have

$$\int_X f_\gamma d\mu \to \int_X f d\mu$$

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For any such net, we have (by the previous paragraphs) that

$$\int_X \bar{f}_\gamma d\mu \to \int_X \bar{f} d\mu \quad \text{and} \quad \int_X f_\gamma g\, d\mu \to \int_X fg\, d\mu$$

for all bounded Borel measurable g.

We now have all the tools to establish the last result of this section.

Theorem 39 *Let X be a compact Hausdorff space, and let ρ be a non-degenerate representation of $C(X)$ on a Hilbert space $\mathcal{H}$. Then there exists a unique projection-valued measure E defined on the Borel subsets of X such that $\rho(f) = \int_X f dE$ for all bounded Borel measurable functions f on X. Moreover, an operator S in $\mathcal{B}(\mathcal{H})$ commutes with $\rho(f)$ for all f in $C(X)$ if and only if S commutes with $E(\omega)$ for every Borel subset ω of X.*

Proof: If the characteristic function χ_ω of a Borel set ω happens to be continuous, we must have $E(\omega) = \int_X \chi_\omega dE = \rho(\chi_\omega)$. In order to make sense of this for a general Borel set ω, we shall extend ρ to a representation $\hat{\rho}$ of the algebra $B(X)$ of all bounded Borel functions on X.

For any pair x and y of vectors in $\mathcal{H}$, the map

$$\tau_{x,y} : f \longrightarrow \langle \rho(f)x, y\rangle$$

is a linear functional on $C(X)$. It is bounded, since

$$|\langle \rho(f)x, y\rangle| \leq \|\rho(f)\|\|x\|\|y\| \leq \|f\|_\infty \|x\|\|y\|$$

for all f in $C(X)$, and in fact $\|\tau_{x,y}\| \leq \|x\|\|y\|$. By the Riesz Representation Theorem, there exists a unique Radon measure $\nu_{x,y}$ on X such that

$$\langle \rho(f)x, y\rangle = \int_X f d\nu_{x,y}$$

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for all f in $C(X)$, and $\|\nu_{x,y}\| = \|\tau_{x,y}\| \le \|x\|\|y\|$. It follows easily from the uniqueness of $\nu_{x,y}$ that the map $(x,y) \to \nu_{x,y}$ is linear in x and conjugate linear in y. Now for a given f in $B(X)$, define

$$\psi_f(x,y) = \int_X f d\nu_{x,y}.$$

Then $|\psi_f(x,y)| \le \|f\|_\infty \|\nu_{x,y}\| \le \|f\|_\infty \|x\|\|y\|$, so ψ_f is a bounded sesquilinear form on $\mathcal{H}$. Thus there exists a unique operator $\hat{\rho}(f)$ in $\mathcal{B}(\mathcal{H})$ such that

$$\langle \hat{\rho}(f)x, y\rangle = \psi_f(x,y) = \int_X f d\nu_{x,y}$$

for all x and y. We claim that $\hat{\rho}$ is a representation of $B(X)$. Note that if $f \in C(X)$, then

$$\langle \hat{\rho}(f)x, y\rangle = \psi_f(x,y) = \int_X f d\nu_{x,y} = \langle \rho(f)x, y\rangle,$$

so that $\hat{\rho} = \rho$ on $C(X)$.

To see that $\hat{\rho}$ is multiplicative, consider f and g in $B(X)$. Consider any net $\{f_\gamma\}$ in $C(X)$ such that $\hat{f}_\gamma \to \hat{f}$ weak* in $C(X)^{**}$, i.e., such that

$$\int_X f_\gamma h d\nu \to \int_X f h d\nu$$

for every measure ν in $M(X)$ and every h in $B(X)$. Then with $\nu = \nu_{x,y}$, we get

$$\langle \hat{\rho}(f_\gamma h)x, y\rangle = \int_X f_\gamma h d\nu_{x,y} \to \int_X f h d\nu_{x,y} = \langle \hat{\rho}(fh)x, y\rangle$$

for every h in $B(X)$ and every x and y in $\mathcal{H}$. This means that for every h in $B(X)$, we have

$$\hat{\rho}(f_\gamma h) \to \hat{\rho}(fh) \quad \text{weak operator.}$$

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In particular,

$$\hat{\rho}(f_\gamma) \to \hat{\rho}(f) \quad \text{and} \quad \hat{\rho}(f_\gamma g) \to \hat{\rho}(fg)$$

in the weak operator topology.

Now suppose, for the moment, that g is continuous on X. Then

$$\hat{\rho}(f_\gamma g) = \rho(f_\gamma g) = \rho(f_\gamma)\rho(g) = \hat{\rho}(f_\gamma)\rho(g),$$

since ρ is multiplicative on $C(X)$. Moreover,

$$\hat{\rho}(f_\gamma)\rho(g) \to \hat{\rho}(f)\rho(g),$$

since multiplication is separately continuous in the weak operator topology. Thus $\hat{\rho}(f_\gamma g)$ converges weak operator to both $\hat{\rho}(fg)$ and $\hat{\rho}(f)\rho(g)$, whence we have

$$\hat{\rho}(fg) = \hat{\rho}(f)\rho(g) = \hat{\rho}(f)\hat{\rho}(g).$$

Also

$$\hat{\rho}(f_\gamma g) = \rho(f_\gamma g) = \rho(g f_\gamma) = \rho(g)\rho(f_\gamma) = \rho(g)\hat{\rho}(f_\gamma) \to \rho(g)\hat{\rho}(f) = \hat{\rho}(g)\hat{\rho}(f)$$

in the weak operator topology. Thus when g is continuous, we also have

$$\hat{\rho}(g)\hat{\rho}(f) = \hat{\rho}(f)\hat{\rho}(g) = \hat{\rho}(fg)$$

and it follows that

$$\hat{\rho}(gf) = \hat{\rho}(g)\hat{\rho}(f)$$

for continuous g. Thus we have

$$\hat{\rho}(fg) = \hat{\rho}(f)\hat{\rho}(g)$$

if either of f or g is continuous.

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For the general case, let f and g be arbitrary in $B(X)$, and choose f_γ in $C(X)$ as above. Then since each f_γ is continuous,

$$\hat{\rho}(f_\gamma g) = \hat{\rho}(f_\gamma)\hat{\rho}(g) \to \hat{\rho}(f)\hat{\rho}(g)$$

weak operator. Since

$$\hat{\rho}(f_\gamma g) \to \hat{\rho}(fg),$$

weak operator, we have $\hat{\rho}(fg) = \hat{\rho}(f)\hat{\rho}(g)$, so $\hat{\rho}$ is indeed multiplicative.

It is easy to check that $\hat{\rho}$ is linear. To see that $\hat{\rho}$ preserves adjoints, choose f_γ in $C(X)$ so that $\hat{f_\gamma} \to \hat{f}$ weak* in $C(X)^{**}$. Then $\hat{\bar{f}}_\gamma \to \hat{\bar{f}}$ weak* and $[\rho(f_\gamma)]^* = \rho(\bar{f}_\gamma)$, so

$$\langle [\rho(f_\gamma)]^* x, y\rangle = \langle \rho(\bar{f}_\gamma)x, y\rangle = \int_X \bar{f}_\gamma d\nu_{x,y} \to \int_X \bar{f} d\nu_{x,y} = \langle \hat{\rho}(\bar{f})x, y\rangle.$$

But $[\rho(f_\gamma)]^* = [\hat{\rho}(f_\gamma)]^* \to [\hat{\rho}(f)]^*$ weak operator, since the adjoint is weak operator continuous. It follows that $[\hat{\rho}(f)]^* = \hat{\rho}(\bar{f})$. This completes the proof that $\hat{\rho}$ is a representation of $B(X)$. □

Now define E by $E(\omega) = \hat{\rho}(\chi_\omega)$, where ω is any Borel subset of X. Since χ_ω is a projection in $B(X)$ and $\hat{\rho}$ is a representation, each $E(\omega)$ is a projection in $\mathcal{B}(\mathcal{H})$. Since the characteristic functions of the empty set and the space X are the constant functions 0 and 1 respectively, and since ρ is non-degenerate, we have $E(\phi) = 0$ and $E(X) = I$. If ω_1 and ω_2 are Borel sets, then $\chi_{\omega_1 \cap \omega_2} = \chi_{\omega_1}\chi_{\omega_2}$, so

$$E(\omega_1 \cap \omega_2) = \hat{\rho}(\chi_{\omega_1 \cap \omega_2}) = \hat{\rho}(\chi_{\omega_1})\hat{\rho}(\chi_{\omega_2}) = E(\omega_1)E(\omega_2)$$

It remains to show that E is countably additive, so suppose $\{\omega_i\}_{i=1}^{+\infty}$ is a pairwise disjoint sequence of Borel subsets of X, and let $\omega = \bigcup_{i=1}^{+\infty} \omega_i$. Then

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for each x in $\mathcal{H}$,

$$\begin{aligned}\langle E(\omega)x, x\rangle = \langle\hat{\rho}(\chi_\omega)x, x\rangle = \int_X \chi_\omega d\nu_{x,x} = \nu_{x,x}(\omega) = \nu_{x,x}\left(\bigcup_{i=1}^{+\infty}\omega_i\right) \\ = \sum_{i=1}^{+\infty}\nu_{x,x}(\omega_i) = \sum_{i=1}^{+\infty}\int_X \chi_{\omega_i} d\nu_{x,x} \\ = \sum_{i=1}^{+\infty}\langle\hat{\rho}(\chi_{\omega_i})x, x\rangle = \sum_{i=1}^{+\infty}\langle E(\omega_i)x, x\rangle,\end{aligned}$$

since each $\nu_{x,x}$ is a measure. It follows that $\sum_{i=1}^{+\infty} E(\omega_i)$ converges weak operator (and hence strong operator) to $E(\omega)$. In particular, E is countable additive, so E is a projection-valued measure.

To see that $\rho(f) = \int_X f dE$ for every continuous f, recall that the projection-valued measure E gives rise to a family of complex measures $E_{x,y}$ such that

$$E_{x,y}(\omega) = \langle E(\omega)x, y\rangle \quad \text{and} \quad \left\langle\left(\int_X f dE\right)x, y\right\rangle = \int_X f dE_{x,y}$$

for all x and y in $\mathcal{H}$ and all bounded Borel functions f on X. From the definitions above we also have

$$\langle\hat{\rho}(f)x, y\rangle = \int_X f d\nu_{x,y}$$

for all such x, y and f. Thus for all Borel sets ω we have

$$E_{x,y}(\omega) = \langle E(\omega)x, y\rangle = \langle\hat{\rho}(\chi_\omega)x, y\rangle = \int_X \chi_\omega d\nu_{x,y} = \nu_{x,y}(\omega),$$

so that $\nu_{x,y} = E_{x,y}$ for all x and y. It follows that

$$\langle\hat{\rho}(f)x, y\rangle = \int_X f d\nu_{x,y} = \int_X f dE_{x,y} = \left\langle\int_X (f dE)x, y\right\rangle$$

for all x and y in $\mathcal{H}$ and all f in $B(X)$. In particular, $\rho(f) = \hat{\rho}(f) = \int_X f dE$ when f is continuous. This completes the proof of the existence of E, and we leave the proof of uniqueness as an exercise.

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Suppose now that S commutes with $E(\omega)$ for all Borel subsets ω of X. Let f be continuous on X, and choose a sequence $\{s_n\}$ of Borel measurable simple functions on X such that $\|s_n - f\|_\infty \to 0$. Every representation of $B(X)$ is norm-decreasing, so we have also that $\|\hat{\rho}(s_n) - \hat{\rho}(f)\| \to 0$. It follows that $\hat{\rho}(f) = \rho(f)$ lies in the closed linear span of the family $\{E(\omega) : \omega$ is a Borel subset of $X\}$. Since S commutes with each element of this family, S commutes with anything in its closed linear span, and in particular, S commutes with $\rho(f)$ for every f in $C(X)$.

Conversely suppose that S commutes with $\rho(f)$ for every f in $C(X)$. Then for all x and y in $\mathcal{H}$ and all f in $C(X)$ we have

$$\int_X f d\nu_{x,S^*y} = \langle \rho(f)x, S^*y\rangle = \langle S\rho(f)x, y\rangle = \langle \rho(f)Sx, y\rangle = \int_X f d\nu_{Sx,y}.$$

By the Riesz Theorem we have $\nu_{x,S^*y} = \nu_{Sx,y}$, so that

$$\begin{aligned}\langle SE(\omega)x, y\rangle = \langle E(\omega)x, S^*y\rangle = \langle \hat{\rho}(\chi_\omega)x, S^*y\rangle &= \int_X \chi_\omega d\nu_{x,S^*y} \\ = \int_X \chi_\omega d\nu_{Sx,y} = \langle \hat{\rho}(\chi_\omega)Sx, y\rangle &= \langle E(\omega)Sx, y\rangle\end{aligned}$$

for all x and y in $\mathcal{H}$ and all Borel sets ω. But then we have $SE(\omega) = E(\omega)S$ for all Borel subsets ω of X, and the proof of the theorem is complete.

Remark 19 The argument in the next to last paragraph above works equally well for any bounded Borel measurable function on X, so it follows that an element S of $\mathcal{B}(\mathcal{H})$ commutes with every $E(\omega)$ if and only if it commutes with $\hat{\rho}(f)$ for every f in $B(X)$.

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The Spectral Theorem and the measurable functional calculus

We are now ready to establish the measurable versions of the spectral theorem and the functional calculus.

Recall that if E is a projection-valued measure on a σ-algebra in X, then the function $f \to \int_X f dE$ is a representation of $B(X)$. On occasion it will be convenient for us to introduce a variable into our notation for the integral, and we may choose to write $\int_X f(\lambda)dE(\lambda)$ instead of $\int_X f dE$.

The point of the next lemma is that two representations agree on $C(sp(T))$ if and only if they agree on an element which generates $C(sp(T))$ as a C^*-algebra.

Lemma 5 *Let T be a normal operator in $\mathcal{B}(\mathcal{H})$, and let E be a projection-valued measure on the Borel subsets of $sp(T)$ with values in $\mathcal{B}(\mathcal{H})$. Then $f(T) = \int_{sp(T)} f(\lambda)dE(\lambda)$ for all f in $C(sp(T))$ if and only if $T = \int_{sp(T)} \lambda dE(\lambda)$.*

Proof: Let $\rho : f \to f(T)$ be the continuous functional calculus, and let $\pi(f) = \int_{sp(T)} f dE$ for f in $C(sp(T))$. Then ρ and π are representations of $C(sp(T))$. The lemma asserts that $\rho = \pi$ on $C(X)$ if and only if $\rho = \pi$ at the identity function $id : \lambda \to \lambda$. Since ρ and π are $*$-homomorphisms, $\rho = \pi$ at id implies that $\rho = \pi$ on every polynomial in z and $\bar{z}$. Since ρ and π are continuous, it follows that $\rho = \pi$ at id implies that $\rho = \pi$ on the closure of these polynomials, i.e., on $C(sp(T))$. The converse is clear. □

Theorem 40 (The Spectral Theorem) *Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Then there exists a unique projection-valued measure E on the Borel subsets of*

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$sp(T)$ such that $T = \int_{sp(T)} \lambda dE(\lambda)$. Moreover,

a) For every f in $C(sp(T))$ we have $f(T) = \int_{sp(T)} f(\lambda) dE(\lambda)$;

b) If ω is a non-empty relatively open subset of $sp(T)$, then $E(\omega) \neq 0$;

c) If $S \in \mathcal{B}(\mathcal{H})$, then $ST = TS$ if and only if $SE(\omega) = E(\omega)S$ for every Borel subset of $sp(T)$ if and only if $S\left(\int_{sp(T)} f dE\right) = \left(\int_{sp(T)} f dE\right) S$ for all bounded Borel functions on $sp(T)$; and

d) The function $f \to \int_{sp(T)} f(\lambda) dE(\lambda)$ is a representation of $B(sp(T))$.

Proof: The continuous functional calculus $\rho : f \to f(T)$ is an isometric representation of $C(sp(T))$ on $\mathcal{H}$ with image $C^*(T)$. By the last result of the previous section there exists a unique projection-valued measure E on the Borel subsets of $sp(T)$ such that a) holds. By the lemma the measure E is actually unique subject to the condition $T = \int_{sp(T)} \lambda dE(\lambda)$. For any projection-valued measure E on the Borel subsets of $sp(T)$, the map $f \to \int_{sp(T)} f(\lambda) dE(\lambda)$ is a representation of $B(sp(T))$. Thus the existence and uniqueness assertions are established, and a) and d) hold.

If ω is a non-empty relatively open subset of $sp(T)$, choose t in ω. Then the complement of ω and the singleton $\{t\}$ are closed in $sp(T)$. By Urysohn's Lemma there exists a continuous function f from $sp(T)$ into $[0, 1]$ which is 0 on the complement of ω and 1 at t. It follows that $0 \leq f \leq \chi_\omega$ and that f is a non-zero element of $C(sp(T))$. Since ρ is injective and representations preserve order, we have $\rho(f) \neq 0$ and $0 \leq \rho(f) = f(T) = \int_{sp(T)} f dE \leq \int_{sp(T)} \chi_\omega dE = E(\omega)$. If $E(\omega)$ were zero, then we would have $\rho(f) = 0$, so b) is established.

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By the Fuglede-Putnam-Rosenblum Theorem, we have $ST = TS$ if and only if $ST = TS$ and $ST^* = T^*S$. Approximating an arbitrary element of $C(sp(T))$ by a polynomial in z and $\bar{z}$ shows that $ST = TS$ if and only if $Sf(T) = f(T)S$ for all f in $C(sp(T))$. By the last result of the previous section, we then have $ST = TS$ if and only if $SE(\omega) = E(\omega)S$ for every Borel subset ω of $sp(T)$. Approximating a bounded Borel function by simple functions then shows that $ST = TS$ if and only if $S\left(\int_{sp(T)} fdE\right) = \left(\int_{sp(T)} fdE\right)S$. □

Definition 29 The unique projection-valued measure E of the last theorem is called the *spectral measure* for T. The *spectral decomposition* of T is the resolution of T into the integral $\int_{sp(T)} \lambda dE(\lambda)$. The association of the integral $\int_{sp(T)} f(\lambda)dE(\lambda)$ with the bounded Borel function f is the *measurable functional calculus* for T. Clearly the measurable functional calculus is an extension of the continuous functional calculus.

Example 8 Let M be a normal matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We identify M with multiplication on C^k for some k. For each i, let E_i be the orthogonal projection onto the eigenspace of λ_i. Then $M = \sum_{i=0}^{n} \lambda_i E_i$, and for each subset ω of $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = sp(M)$, $E(\omega) = \sum_{\lambda_i \in \omega} E_i$. Note that $E(\lambda_i)$ is non-zero for each $i = 1, 2, \ldots, n$.

Example 9 Let T in $\mathcal{B}(L^2[0,1])$ be defined by $(Tf)(t) = tf(t)$, where $t \in [0,1]$. Then $sp(T) = [0,1]$, and $E(\omega)f = \chi_\omega f$ for all f in $L^2[0,1]$ and all Borel subsets of $[0,1]$ (exercise). Thus for any f and g in $L^2[0,1]$ and any

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Borel set ω we have

$$\langle E(\omega)f, g\rangle = E_{f,g}(\omega) = \int_0^1 \chi_\omega f\bar{g} = \int_\omega f\bar{g}.$$

In particular, if ω has Lebesgue measure zero, then $E(\omega) = 0$. Conversely, if $E(\omega) = 0$, then $0 = \langle E(\omega)1, 1\rangle = \int_0^1 \chi_\omega$, so that the Lebesgue measure of ω is zero.

Remark 20 The continuous functional calculus on $C(sp(T))$ is always isometric, but the last example shows that the measurable functional calculus is not injective. Since the measurable calculus is also a representation, we have

$$\begin{aligned}
\int_{sp(T)} f dE = 0 &\quad \text{if and only if} \\
\int_{sp(T)} |f|^2 dE = \left(\int_{sp(T)} f dE\right)^* \int_{sp(T)} f dE = 0 &\quad \text{if and only if} \\
\int_{sp(T)} |f|^2 dE_{x,x} = 0 &\quad \text{for all } x \text{ if and only if} \\
f = 0 &\quad \text{almost everywhere with respect to every } E_{x,x}.
\end{aligned}$$

In particular,

$$E(\omega) = 0 \quad \text{if and only if} \quad E_{x,x}(\omega) = 0 \quad \text{for all } x.$$

It can be shown (see for example [KRII, Thm. 10.1.7]) that if we identify two functions that agree almost everywhere with respect to every $E_{x,x}$, (i.e., if we factor out the kernel of the measurable functional calculus) then we get a quotient C^*-algebra on which map induced by $f \to \int_{sp(T)} f dE$ is isometric.

The measurable functional calculus is well-behaved with respect to pointwise convergence of sequences in $B(sp(T))$.

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Theorem 41 *Let $\{f_n\}$ be a bounded sequence in $B(sp(T))$, and suppose that $\{f_n\}$ converges pointwise on $sp(T)$ to a function f. Then $\int_{sp(T)} f_n dE \to \int_{sp(T)} f dE$ weak operator.*

Proof: If each f_n is real-valued, then for all x in H, the dominated convergence theorem implies that

$$\left\langle \left(\int_{sp(T)} f_n dE \right) x, x \right\rangle = \int_{sp(T)} f_n dE_{x,x} \to \int_{sp(T)} f dE_{x,x}$$
$$= \left\langle \left(\int_{sp(T)} f dE \right) x, x \right\rangle .$$

To get the general case, we take real and imaginary parts. □

Corollary 42 *Let $\{f_n\}$ be a non-decreasing sequence of non-negative functions in $B(sp(T))$, and suppose that $\{f_n\}$ converges pointwise on $sp(T)$ to a function f in $B(sp(T))$. Then $\int_{sp(T)} f_n dE \to \int_{sp(T)} f dE$ weak operator.*

In particular, for bounded non-decreasing sequences in $B(sp(T))$, the measurable functional calculus takes the least upper bound of $\{f_n\}$ in $B(sp(T))$ to the least upper bound of the sequence $\left\{ \int_{sp(T)} f_n dE \right\}$ in $\mathcal{B}(\mathcal{H})$.

Example 10 Let T be a positive operator in $\mathcal{B}(\mathcal{H})$, so that $sp(T) \subseteq [0, +\infty)$. Let χ be the characteristic function of $(0, +\infty)$, and let $f_n(t) = \sqrt[n]{t}$ for $t \in sp(T)$. Then $f_n \to \chi$ pointwise on $sp(T)$, and the sequence $\{f_n\}$ is bounded on $sp(T)$. By the theorem, we have $\int_{sp(T)} f_n dE \to \int_{sp(T)} \chi dE$ weak operator. Now $\int_{sp(T)} f_n dE = T^{1/n}$, and we saw in an earlier section that $T^{1/n}$ converges weak operator to the orthogonal projection E_T of $\mathcal{H}$ onto the range of T. Thus the functional calculus takes χ to the range projection E_T

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of T. Thus, if we were to write $f(T)$ for the measurable functional calculus applied to f (this is often done), then we would have

$$\chi(T) = E_T$$

for any positive T.

Example 9 revisited Note that when (as above) $\mathcal{H} = L^2[0,1]$ and $Tf(t) = tf(t)$ for t in $[0,1]$, the range of T is actually dense in $\mathcal{H}$, so that

$$\int_{sp(T)} \chi = \chi(T) = I = \int_{sp(T)} 1 = 1(T),$$

i.e.,

$$E((0,1]) = I = E([0,1]),$$

from which we get again the fact that the singleton set $\{0\}$ is a null set for E. In the next section we shall see that for any normal T, a singleton $\{\lambda\}$ in $sp(T)$ is *not* a null set for E if and only if λ is actually an eigenvalue of T.

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Some applications of the measurable functional calculus

Eigenvalues and eigenspaces Suppose T is a normal operator. Using the continuous functional calculus, we saw in a previous section that an isolated point in the spectrum of T is always an eigenvalue of T. With the measurable calculus we can give a characterization of the eigenvalues of T.

Theorem 42 *Let T be a normal element of $\mathcal{B}(\mathcal{H})$, and let E be the spectral measure of T. Let λ_0 be an element of $sp(T)$. Then λ_0 is an eigenvalue of T if and only if $E(\{\lambda_0\}) \neq 0$, in which case $E(\{\lambda_0\})$ is the orthogonal projection of $\mathcal{H}$ onto the eigenspace of λ_0.*

Proof: For each positive integer n, let $\omega_n = \left\{\lambda \in sp(T) : |\lambda - \lambda_0| > \frac{1}{n}\right\}$, and let

$$g_n(\lambda) = \frac{1}{\lambda - \lambda_0} \quad \text{for } \lambda \in \omega_n \text{ and } g_n(\lambda) = 0 \text{ otherwise.}$$

Then each g_n is a bounded Borel function on $sp(T)$, and $g_n(\lambda)(\lambda - \lambda_0) = \chi_{\omega_n}(\lambda)$ for all n and all $\lambda \in sp(T)$. Since $h \to \int_{sp(T)} hdE$ is a representation on $B(sp(T))$, we have for all n that

$$\int_{sp(T)} g_n(\lambda)dE(\lambda) \int_{sp(T)} \lambda - \lambda_0 dE(\lambda) = \int_{sp(T)} \chi_{\omega_n}(\lambda)dE(\lambda),$$

i.e.,

$$\left(\int_{sp(T)} g_n dE\right)(T - \lambda_0 I) = E(\omega_n).$$

Since $\omega_n \subseteq \omega_{n+1}$ for all n, it follows from the countable additivity of E that the sequence $\{E(\omega_n)\}$ converges monotonically up in the strong operator topology to $I - E(\{\lambda_0\}) = E\left(\bigcup_1^{+\infty} \omega_n\right)$.

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If now $(T - \lambda_0 I)y = 0$, then $E(\omega_n)y = \left(\int_{sp(T)} g_n dE\right)(T - \lambda_0 I)y = 0$ for all n, so that $(I - E(\{\lambda_0\})y = 0$, i.e., $y = E(\{\lambda_0\})y$. In particular, if λ_0 is an eigenvalue of T, then $E(\{\lambda_0\}) \neq 0$, and the eigenspace of λ_0 is contained in the range of $E(\{\lambda_0\})$.

Conversely, suppose that $E(\{\lambda_0\}) \neq 0$. Let f_0 be the characteristic function of $\{\lambda_0\}$, so that $\int_{sp(T)} f_0 dE = E(\{\lambda_0\}) \neq 0$. Now $(\lambda - \lambda_0)f_0(\lambda) = 0$ for all λ, so that

$$0 = (T - \lambda_0 I) \int_{sp(T)} f_0 dE = (T - \lambda_0 I)E(\{\lambda_0\}).$$

For each y in the range of $E(\{\lambda_0\})$, we have

$$(T - \lambda_0 I)y = (T - \lambda_0 I)E(\{\lambda_0\})y = 0.$$

In particular, the non-zero elements of the range of $E(\{\lambda_0\})$ are eigenvectors for T and λ_0, and the eigenspace of λ_0 contains the range of $E(\{\lambda_0\})$. □

Corollary 43 *Let $T \in \mathcal{B}(\mathcal{H})$ be normal, and let E be the spectral measure of T. Let χ be the characteristic function of $sp(T)\backslash\{0\}$, and let E_T be the orthogonal projection of $\mathcal{H}$ onto the closure of the range of T. Then:*

a) $E(\{0\})$ *is the orthogonal projection of $\mathcal{H}$ onto the kernel of T; and*

b) $\int_{sp(T)} \chi dE = E(sp(T)\backslash\{0\}) = E_T$.

Proof: a) If $\ker(T) = \{0\}$, then zero is not an eigenvalue of T, so that $E(\{0\}) = 0$. But then $E(\{0\})$ is the projection of $\mathcal{H}$ onto $\ker(T)$. So suppose $\ker(T) \neq \{0\}$. Then $E(\{0\})$ is the projection onto the eigenspace of zero, i.e., onto the kernel of T. In either case, part a) is established.

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b) Since $sp(T)$ is the disjoint union of $\{0\}$ and $sp(T)\backslash\{0\}$, we have $I = E(sp(T)) = E(\{0\}) + E(sp(T)\backslash\{0\})$, and $E(sp(T)\backslash\{0\})$ is the projection of $\mathcal{H}$ onto $E(\{0\})^{\perp} = \ker(T)^{\perp}$. Since T is normal, we have $\|Tx\| = 0$ if and only if $\|T^*x\| = 0$, so $\ker T = \ker T^*$. But then $\ker(T)^{\perp} = \ker(T^*)^{\perp} = \overline{R(T)}$, so that $\int_{sp(T)} \chi dE = E(sp(T)\backslash\{0\}) = E_T$. $\square$

Exercise (An alternative proof that $\int_{sp(T)} \chi dE = E_T$) *Let $T \in \mathcal{B}(\mathcal{H})$ be normal, with spectral measure E. For each $n \geq 1$, let $f_n(z) = \min\{|z|n, 1\}$.*

a) *Show that $\{f_n\}$ converges monotonically on $\mathbb{C}$ to the characteristic function χ of $\mathbb{C}\backslash\{0\}$, and conclude that $\int_{sp(T)} f_n dE$ converges strong operator to $\int_{sp(T)} \chi dE$.*

b) *Show that $zf_n(z)$ converges uniformly on $\mathbb{C}$ to z, and hence that $Tf_n(T)$ converges in norm to T. Conclude from a) that $T\chi(T) = \chi(T)T = T$, where $\chi(T) = \int_{sp(T)} \chi dE$, and that $R(\chi(T)) \supseteq \overline{R(T)}$.*

c) *Use the Weierstrass Approximation Theorem to show that $R(\chi(T)) \subseteq \overline{R(T)}$. Conclude that $\chi(T) = E_T$.*

Note that in the last corollary and exercise, $E_T = I$ if and only if T is injective.

The numerical range and the norm The *numerical range* of a bounded linear operator T is the set

$$\{\langle Tx, x\rangle : x \in H, \|x\| = 1\}.$$

For normal operators T, the norm of T is determined by the numerical range.

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Theorem 43 *Let T be a normal element of $\mathcal{B}(\mathcal{H})$. Then $\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$.*

Proof: Since T is normal, its norm coincides with its spectral radius, so there exists λ_0 in $sp(T)$ such that $\|T\| = |\lambda_0|$. Now let $\epsilon > 0$, and let

$$\omega = \{\lambda \in sp(T) : |\lambda - \lambda_0| < \epsilon\}.$$

Then ω is relatively open, so $E(\omega) \neq 0$. Let

$$f(\lambda) = (\lambda - \lambda_0)\chi_\omega(\lambda),$$

so that

$$f(T) = (T - \lambda_0 I)E(\omega).$$

If x is any unit vector from the range of $E(\omega)$, then

$$f(T)x = (T - \lambda_0 I)E(\omega)x = (T - \lambda_0 I)x,$$

so that

$$\begin{aligned} |\langle Tx, x\rangle - \lambda_0| &= |\langle Tx, x\rangle - \lambda_0\langle x, x\rangle| \\ &= |\langle (Tx - \lambda_0 I)x, x\rangle| = |\langle f(T)x, x\rangle| \leq \|f(T)\| < \epsilon. \end{aligned}$$

One can also show (see []) that the closure of the numerical range is the convex hull of the spectrum. □

Compact operators The next theorem is the key to an explicit description of the spectral theory of a compact normal operator.

Theorem 44 *Let T be a normal element of $\mathcal{B}(\mathcal{H})$, and let E be the spectral measure of T. Then T is compact if and only if for every $\epsilon > 0$, $E(\{\lambda : |\lambda| > \epsilon\})$ is of finite rank.*

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Proof: Suppose that for every $\epsilon > 0$, $E(\{\lambda : |\lambda| > \epsilon\})$ is of finite rank. Let $\omega_n = \left\{\lambda : |\lambda| > \frac{1}{n}\right\}$. Then for every n, $E(\chi_{\omega_n})T = TE(\chi_{\omega_n})$ is of finite rank. Since $f \to \int_{sp(T)} f dE$ is a representation of $B(sp(T))$,

$$TE(\chi_{\omega_n}) = \int_{sp(T)} \lambda dE(\lambda) \int_{sp(T)} \chi_{\omega_n}(\lambda) dE(\lambda) = \int_{sp(T)} \lambda \chi_{\omega_n}(\lambda) dE(\lambda).$$

Thus

$$T - TE(\chi_{\omega_n}) = \int_{sp(T)} \lambda - \lambda\chi_{\omega_n}(\lambda) dE(\lambda) = \int_{sp(T)} f_n(\lambda) dE(\lambda),$$

where $f_n(\lambda) = \lambda - \lambda\chi_{\omega_n}(\lambda)$. It follows that

$$\|T - TE(\chi_{\omega_n})\| = \left\| \int_{sp(T)} f_n(\lambda) dE(\lambda) \right\| \leq \|f_n\|_\infty,$$

since representations are norm-decreasing. But $\|f_n\|_\infty \leq \frac{1}{n}$, so T is a norm limit of finite rank operators, and hence is compact. □

Conversely, suppose there exists $\epsilon > 0$ such that the range of $E(\{\lambda : |\lambda| > \epsilon\})$ is infinite dimensional. Let $\omega = \{\lambda : |\lambda| > \epsilon\}$, and let $f(\lambda) = \frac{1}{\lambda}\chi_\omega(\lambda)$. Then $\lambda f(\lambda) = \chi_\omega(\lambda)$, so $T\left(\int_{sp(T)} f dE\right) = E(\omega)$. Since the range of $E(\omega)$ is infinite dimensional, $E(\omega)$ is not compact. But the compact operators form an ideal in $\mathcal{B}(\mathcal{H})$, so it follows that T cannot be compact.

Theorem 45 *Let T be a compact normal operator in a $\mathcal{B}(\mathcal{H})$. Then:*

a. *Every non-zero element of the spectrum of T is an eigenvalue of T;*

b. *If λ is a cluster point of $sp(T)$, then $\lambda = 0$;*

c. *$sp(T)$ is countable;*

d. *For each non-zero λ in $sp(T)$, the eigenspace of λ is finite dimensional;*

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e. If E is the spectral measure of T and $E_\lambda = E(\{\lambda\})$, then the spectral decomposition of T takes the form

$$T = \sum_{\lambda \in sp(T)} \lambda E_\lambda$$

f. There exists an orthonormal basis $\{e_\gamma\}$ for $\mathcal{H}$ consisting of eigenvectors for T, and all but countably many of the e_γ lie in the kernel of T.

Proof: Exercise. □

Corollary 44 *If $\mathcal{H}$ is separable and $T \in \mathcal{B}(\mathcal{H})$ is compact and normal, then there exists a countable orthonormal basis $\{e_n\}$ for $\mathcal{H}$ and a family $\{\lambda_n\}$ of scalars (not necessarily distinct) such that for every x in $\mathcal{H}$ we have*

$$T = \sum_{n=1}^{+\infty} \lambda_n \langle x, e_n \rangle e_n.$$

We can also give a complete description of the closed ideals in $\mathcal{B}(\mathcal{H})$, at least for separable $\mathcal{H}$.

Proposition 54 *Let J be an ideal in $\mathcal{B}(\mathcal{H})$. If $J \neq 0$, then J contains all finite rank operators. If J is closed and $J \neq 0$, then J contains all compact operators.*

Proof: The second assertion follows from the first, since the compact operators are the closure of the finite rank operators. If the range of T is finite dimensional, choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for it, and we have $Tx = \sum_{i=1}^{n} \langle Tx, e_k \rangle e_k$ for all x in $\mathcal{H}$. Thus each finite rank operator is a sum of rank one operators $x \to \langle Tx, e_k \rangle e_k$, so it suffices to show that J contains all rank one operators. If T is a rank one operator and E is the orthogonal

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projection onto the range of T, then $ET = T$. Thus any ideal which contains all rank one projections contains all rank one operators, so it suffices to show that if E is a rank one projection and J is a non-zero ideal in $\mathcal{B}(\mathcal{H})$, then $E \in J$. To do this, choose $T \neq 0$ in J. Suppose $Tx_1 = x_2 \neq 0$, so that $x_1 \neq 0$. Then define A and B in $\mathcal{B}(\mathcal{H})$ as follows. Let x_0 be a non-zero vector in the range of E. Put $Ax_0 = x_1$, put $A = 0$ on $\{x_0\}^\perp$, and extend A by linearity to all of $\mathcal{H}$. Put $Bx_2 = x_0$, put $B = 0$ on $\{x_2\}^\perp$, and extend B by linearity to $\mathcal{H}$. Then $BTAx_0 = BTx_1 = Bx_2 = x_0 = Ex_0$, so $BTA = E$ on $\text{Range}(E) = \text{span}\{x_0\}$. Also $BTA = E$ on the orthogonal complement of the span of $\{x_0\}$, since $A = 0$ on this orthogonal complement. Since $T \in J$ and J is an ideal, $E = BTA \in J$, and the proof is complete. □

Lemma 6 *Let T be a compact positive element of $\mathcal{B}(\mathcal{H})$. Then $T^{1/2}$ is also compact. Moreover, if $T \in \mathcal{B}(\mathcal{H})$ and T^*T is compact, then T is also compact.*

Proof: Exercise: Use the Weierstrass Approximation Theorem and the polar decomposition of an arbitrary T in $\mathcal{B}(\mathcal{H})$. (Question: How much generalization will your argument support?) □

Theorem 46 *Let $\mathcal{H}$ be a separable Hilbert space, and let J be a closed ideal in $\mathcal{B}(\mathcal{H})$. If J contains a non-zero non-compact operator, then $J = \mathcal{B}(\mathcal{H})$. In particular, if $\mathcal{H}$ is separable, then the only closed ideals in $\mathcal{B}(\mathcal{H})$ are $\{0\}$, $\mathcal{B}(\mathcal{H})$, and the compact operators.*

Proof: Let $T \in J$ be non-zero and non-compact. Then T^*T belongs to J, is normal and non-zero, and (by the lemma) is not compact. Thus we may assume that T is normal. Let E be the spectral measure for T. Then there

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exists $\epsilon > 0$ such that for the set $\omega = \{\lambda : |\lambda| > \epsilon\}$, $E(\omega)$ has infinite rank. Let $f(\lambda) = \frac{1}{\lambda}\chi_\omega(\lambda)$. Then

$$\begin{aligned} T\int_{sp(T)} f dE &= \int_{sp(T)} \lambda dE(\lambda) \int_{sp(T)} f(\lambda) dE(\lambda) \\ &= \int_{sp(T)} \lambda f(\lambda) dE(\lambda) \\ &= \int_{sp(T)} \chi_\omega(\lambda) dE(\lambda) = E(\omega) \end{aligned}$$

belongs to J and has infinite dimensional range. Since $\mathcal{H}$ is separable, $E(\omega)$ and $\mathcal{H}$ have the same (Hilbert space) dimension, so there exists a partial isometry V with initial space equal to the range of $E(\omega)$ and final space equal to $\mathcal{H}$. Then $VE(\omega)V^* = I$ belongs to J, so J is all of $\mathcal{B}(\mathcal{H})$. □

Corollary 45 *Let A denote the compact operators on a separable Hilbert space $\mathcal{H}$. Then any non-zero continuous algebra homomorphism of A into another algebra over $\mathbb{C}$ is injective. Any $*$-homomorphism of $\mathcal{B}(\mathcal{H})$ into a C^*-algebra with identity must either vanish on A or have an isometric restriction to A.*

Proof: The first assertion follows from Theorem 46, and the second follows from Theorem 30 and the first. □

Exercises

1. Carry out the alternate proof (see the text above) that $\int_{sp(T)} \lambda\, dE = E_T$.

2. Prove Theorem 45.

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3. Let A be the C^*-algebra of all operators in $\mathcal{B}(\mathcal{H})$ of the form $T + \lambda I$, where T is compact and $\lambda \in \mathbb{C}$. (See the exercise set for "Ideals and Characters" above.) Show that the function $T + \lambda I \to \lambda$ is the unique character on A.

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The self-adjoint case

Many of the most useful applications of the functional calculus involve self-adjoint, or even positive, operators. When T is self-adjoint, the spectrum of T is real, and we can establish a representation for T by means of a more elementary integral. The key to this representation is a one-parameter family $\{E_\lambda : \lambda \in \mathbf{R}\}$ of projections obtained from the spectral measure of T.

Definition 30 Let T be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$, and let $E : \omega \to E(\omega)$ be its spectral measure. For each real number λ, let E_λ be the projection $E(sp(T) \cap (-\infty, \lambda])$.

Observe that $E_\lambda = 0$ for all $\lambda < \inf sp(T)$ and $E_\lambda = I$ for all $\lambda \geq \sup sp(T)$. Moreover, if $\lambda \leq \mu$, then $E_\lambda \leq E_\mu$. In particular, for each x in $\mathcal{H}$, the function

$$\lambda \to \langle E_\lambda x, x \rangle$$

is non-negative and monotone non-decreasing. It now follows easily from the polarization identity that for each x and y in $\mathcal{H}$, the function

$$h_{x,y} : \lambda \to \langle E_\lambda x, y \rangle$$

is a linear combination of monotone functions. Finally, we note that on each interval in the complement of the spectrum of T, the function $\lambda \to E_\lambda$ is constant.

For the remainder of this section, T will be self-adjoint and m and M will denote respectively the infimum and the supremum of $sp(T)$.

Lemma 7 *Let f be continuous on $sp(T)$, and let g_1 and g_2 be continuous extensions of f to the interval $[m, M]$. Then for all x and y in $\mathcal{H}$,*

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$\int_m^M g_1(\lambda)dh_{x,y}(\lambda) = \int_m^M g_2(\lambda)dh_{x,y}(\lambda)$. *(That is, the Riemann-Stieltjes integral of g with respect to the function $\lambda \to \langle E_\lambda x, y\rangle$ is independent of the choice of the continuous extension g.)*

Proof: This follows from the fact that $\lambda \to E_\lambda$ is constant on each interval in the complement of $sp(T)$. $\square$

For any function in $C(sp(T))$ and any x and y in $\mathcal{H}$, we now take the integral $\int_{sp(T)} f(\lambda)d\langle E_\lambda x, y\rangle$ to be the Riemann-Stieltjes integral $\int_m^M g(\lambda)dh_{x,y}(\lambda)$, where g is any continuous extension of f to $[m, M]$. We see in the next theorem that $\int_{sp(T)} f(\lambda)d\langle E_\lambda x, y\rangle$ coincides with the integral $\int_{sp(T)} f dE_{x,y}$ of f with respect to the complex measure $E_{x,y}$.

Theorem 47 *Let T be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$. Then there exists a one-parameter family $\{E_\lambda\}$ of orthogonal projections such that*

a) $E_\mu E_\lambda = E_\lambda E_\mu = E_\lambda$ *whenever* $\lambda \leq \mu$;

b) $E_\lambda x = \lim_{\epsilon \to 0+} E_{\lambda+\epsilon} x$ *for all real* λ *and all* x *in* $\mathcal{H}$;

c) $E_\lambda = 0$ *for all* $\lambda < \inf sp(T)$ *and* $E_\lambda = I$ *for all* $\lambda \geq \sup sp(T)$;

d) *an operator S in $\mathcal{B}(\mathcal{H})$ commutes with T if and only if S commutes with all the projections E_λ; and*

e) *for any continuous function f on $sp(T)$ and any x and y in $\mathcal{H}$, we have*

$$\langle f(T)x, y\rangle = \int_{sp(T)} f(\lambda)d\langle E_\lambda x, y\rangle.$$

In particular, we have

$$\langle Tx, y\rangle = \int_{sp(T)} \lambda d\langle E_\lambda x, y\rangle = \int_{-\infty}^{+\infty} \lambda d\langle E_\lambda x, y\rangle = \int_m^M \lambda d\langle E_\lambda x, y\rangle,$$

where $m = \inf sp(T)$ *and* $M = \sup sp(T)$.

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Proof: We let E_λ be as above. We have already seen that a) and c) hold. To establish b), let λ_n converge down to λ, and let χ_n and χ denote respectively the characteristic functions of $[0, \lambda_n]$ and $[0, \lambda]$. Then χ_n converges pointwise down on $\mathbf{R}$ to χ, so

$$\begin{aligned}\langle E_{\lambda_n} x, x\rangle = \langle E(sp(T) \cap (-\infty, \lambda_n])x, x\rangle &= \int_{sp(T)} \chi_n dE_{x,x} \downarrow \int_{sp(T)} \chi dE_{x,x} \\ &= \langle E(sp(T) \cap (-\infty, \lambda])x, x\rangle = \langle E_\lambda x, x\rangle\end{aligned}$$

for every x in $\mathcal{H}$, by the Monotone Convergence Theorem. Thus E_{λ_n} converges weak operator to E_λ, and it follows readily that E_λ is the weak operator limit of $E_{\lambda+\epsilon}$. Since the family $\{E_\lambda\}$ is monotone in λ, $E_{\lambda+\epsilon}$ also converges strong operator to E_λ, and b) is established.

To establish d), let $S \in \mathcal{B}(\mathcal{H})$. We need only show that if S commutes with each E_λ, then S commutes with $E(\omega)$ for every Borel subset ω of $sp(T)$. Now for each real λ and μ with $\lambda < \mu$, we have $E_\mu - E_\lambda = E(sp(T) \cap (\lambda, \mu])$, so S commutes with $E(\omega)$ when ω has the form $sp(T) \cap (\lambda, \mu]$. It is easy to check that the set of all Borel subsets ω of $sp(T)$ such that S commutes with $E(\omega)$ is a σ-algebra, and that the subsets of the form $sp(T) \cap (\lambda, \mu])$ generate the σ-algebra of all Borel subsets of $sp(T)$, so S must commute with $E(\omega)$ for every Borel subset ω of $sp(T)$ as required.

Finally, to establish e), it suffices by polarization to consider the case when $x = y$. Let $\epsilon > 0$ be given, and let g be any continuous extension of f to $[m, M]$. Since g is uniformly continuous, we may choose $\delta > 0$ such that $|s - t| < \delta$ implies that $|g(s) - g(t)| < \frac{\epsilon}{2\|x\|^2}$. We consider any partition P of $[m, M]$ into subintervals each of whose width is less than δ, and we denote the endpoints of these subintervals by $\lambda_0, \lambda_1, \ldots, \lambda_n$. We have for all choices

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of intermediate points t_i that

$$\left\| \int_{sp(T)} f dE - \sum_{j=1}^{n} f(t_j) E(sp(T) \cap (\lambda_{j-1} - \lambda_j]) \right\| \leq \frac{\epsilon}{2\|x\|^2} ,$$

so that

$$\left\| \int_{sp(T)} f dE_{x,x} - \sum_{j=1}^{n} f(t_j) \langle (E(sp(T) \cap (\lambda_{j-1} - \lambda_j])x, x \rangle \right\| \leq \frac{\epsilon}{2} .$$

On the other hand, for all choices of t_i we have

$$\sum_{j=1}^{n} f(t_j) E(sp(T) \cap (\lambda_{j-1}, \lambda_j]) = \sum_{j=1}^{n} g(t_j)(E_{\lambda_j} - E_{\lambda_{j-1}}),$$

so that

$$\sum_{j=1}^{n} f(t_j) \langle E(sp(T) \cap (\lambda_{j-1}, \lambda_j])x, x \rangle) = \sum_{j=1}^{n} g(t_j)(\langle E_{\lambda_j} x, x \rangle - \langle E_{\lambda_{j-1}} x, x \rangle)$$

Passing to a refinement of P, we may assume that the Riemann-Stieltjes sums $\sum_{j=1}^{n} g(t_j)(\langle E_{\lambda_j} x, x\rangle - \langle E_{\lambda_{j_{j-1}}} x, x \rangle)$ lie within $\frac{\epsilon}{2}$ of the integral $\int_m^M g(\lambda) d\langle E_\lambda x, x \rangle$. It follows then that

$$\langle f(T)x, x \rangle = \int_{sp(T)} f dE_{x,x} = \int_m^M g(\lambda) d\langle E_\lambda x, x \rangle = \int_{sp(T)} f(\lambda) d\langle E_\lambda x, x \rangle,$$

and the proof is complete. □

Note that by this theorem, the one-parameter family $\{E_\lambda\}$ determines the operator T and its continuous functional calculus. It follows from constructions in our previous sections that the family $\{E_\lambda\}$ completely determines the spectral measure E of T, and hence the measurable functional calculus for T, as well.

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Normal Operators as Multiplication Operators: yet another version of the spectral theorem

Let M be a normal n by n complex matrix, let T be the corresponding normal operator on the Hilbert space $\mathbf{C}^n$, and let Λ be the set of (distinct) eigenvalues of M. We have seen that Λ is also the spectrum of T. Moreover, if we write E_λ for the spectral projection $E(\{\lambda\})$, then E_λ is the projection of $\mathbf{C}^n$ onto the eigenspace of $\mathbf{C}^n$ corresponding to λ, and it is easy to see that $T = \sum_{\lambda\in\Lambda} \lambda E_\lambda$. Indeed, for any vector x in $\mathbf{C}^n$, we may write x as the sum $x = \sum_{\lambda\in\Lambda} x_\lambda$ of its projections $x_\lambda = E_\lambda x$ onto the ranges of the E_λ, and T acts on x by multiplying each component x_λ by λ:

$$Tx = T\left(\sum_{\lambda\in\Lambda} x_\lambda\right) = \sum_{\lambda\in\Lambda} Tx_\lambda = \sum_{\lambda\in\Lambda} \lambda x_\lambda.$$

Thus T acts on the underlying space $\mathbf{C}^n$ as a multiplication operator, carrying each vector in the eigenspace E_λ into a fixed multiple (by λ) of itself. It follows then that $T = \sum_{\lambda\in\Lambda} \lambda E_\lambda$ as claimed, and that we may regard T in a natural way as a multiplication operator.

Let us now choose an orthonormal basis $\mathbf{v} = \{v_1, \ldots, v_n\}$ for $\mathbf{C}^n$ consisting of eigenvectors for M. For each v_j, let λ_j be the eigenvalue of M that corresponds to v_j, and form the vector $(\lambda_1, \ldots, \lambda_n)$ in $\mathbf{C}^n$. For each x in $\mathbf{C}^n$, let $(x_1, \ldots, x_n)$ be the vector of coordinates of x with respect to $\mathbf{v}$. Then T takes x into the vector with $\mathbf{v}$-coordinates $(\lambda_1 x_1, \ldots, \lambda_n x_n)$. If we think of the elements of $\mathbf{C}^n$ as functions on $\{1, 2, \ldots, n\}$, then the action of T on x is equivalent to multiplying the function $(x_1, \ldots, x_n)$ pointwise by the function $(\lambda_1, \ldots, \lambda_n)$. In this section we shall see that any normal

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operator can be realized as multiplication by a measurable function on an appropriately chosen L^2-space. We need some definitions.

Definition 31 Let T be an element of $\mathcal{B}(\mathcal{H})$. and let S be a subspace of $\mathcal{H}$. We say that S is *invariant for* T if $T(S) \subseteq S$. We say that S is a *reducing subspace for* T if S is invariant for T and for T^*.

Note that if S is invariant under T, then the closure of S is also invariant under T. It is easy to check that if S is a closed reducing subspace for a normal T, then the restriction of T to S is a normal bounded linear operator on the Hilbert space S, with adjoint equal to the restriction of T^*.

Proposition 55 *Let T be an element of $\mathcal{B}(\mathcal{H})$, let S be a closed subspace of $\mathcal{H}$, and let E be the orthogonal projection of $\mathcal{H}$ onto S. Then*

1) *S is invariant for T if and only if $ETE = TE$;*
2) *S is reducing for T if and only if $ET = TE$;*
3) *S is invariant for T if and only if $S^\perp$ is invariant for T^*;*
4) *S is reducing for T if and only if $S^\perp$ is reducing for T; and*
5) *S is reducing for T if and only if S and $S^\perp$ are both invariant for T.*

Proof: Exercise. □

Definition 32 Let A be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. A vector x of $\mathcal{H}$ is a *cyclic vector* for A if $\{Tx : T \in A\}$ is dense in $\mathcal{H}$. We say that x is a *cyclic vector* for a normal operator T if x is cyclic for the $*$-algebra $C^*(T)$.

When A is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ and x is a vector in $\mathcal{H}$, we shall often write Ax for the subspace $\{Tx : T \in A\}$ of $\mathcal{H}$ and $[Ax]$ for the closure of

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Ax. Thus x is cyclic for A if and only if Ax is dense in $\mathcal{H}$, i. e., if and only if $[Ax] = \mathcal{H}$. Now suppose A contains the identity operator, so that $x \in Ax$. It is easy to see that Ax is the smallest subspace of $\mathcal{H}$ that contains x and is invariant under every element of A, and that $[Ax]$ is the smallest closed subspace of $\mathcal{H}$ that contains x and is invariant under every element of A.

Now let T be normal in $\mathcal{B}(\mathcal{H})$. Since the algebra $A_0 = \{p(T, T^*) : p$ is a polynomial in two variables$\}$ is norm-dense in $C^*(T)$, A_0x is norm-dense in the subspace $C^*(T)x$ for every x. Thus for every vector x, A_0x and $C^*(T)x$ have the same closure in $\mathcal{H}$, so x is cyclic for T if and only if A_0x is dense in $\mathcal{H}$. It follows easily that $[C^*(T)x]$ is the smallest closed subspace of $\mathcal{H}$ that contains x and is invariant under T and under T^*, and that x is cyclic for T if and only if the smallest closed subspace containing x which is invariant under T and T^* is $\mathcal{H}$.

Example 11 Let T be the operator on $L^2[0, 1]$ given by multiplication by $f(z) = z$. Then the constant function 1 is a cyclic vector for T. Indeed, if p is a polynomial in two variables, then $p(z, \bar{z}) = p(T, T^*)1$. Since the space of all functions of the form $p(z, \bar{z})$ is norm-dense in $C[0, 1]$, it is also dense in $L^2[0, 1]$. (Recall that $C[0, 1]$ is dense in $L^2[0, 1]$.)

Theorem 48 *Let T be a normal operator in $\mathcal{B}(\mathcal{H})$, and suppose that T has a cyclic vector x_0. Let ρ be the continuous functional calculus for T. Then there exist a Radon probability measure μ on $sp(T)$ and an isometry U from $L^2(sp(T), \mu)$ onto $\mathcal{H}$ such that for every f in $C(sp(T)), U^{-1}\rho(f)U$ is multiplication on $L^2(sp(T), \mu)$ by f.*

Proof: We may assume that x_0 is a unit vector. We define a linear functional

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ψ on $C(sp(T))$ by

$$\psi(f) = \langle \rho(f)x_0, x_0 \rangle.$$

Then ψ is bounded and has norm 1, since

$$|\psi(f)| \leq \|\rho(f)x_0\|\|x_0\| \leq \|\rho(f)\|\|x_0\|^2 \leq \|f\|_\infty\,,$$

and

$$\psi(1) = \langle \rho(1)x_0, x_0 \rangle = \langle Ix_0, x_0 \rangle = \|x_0\|^2 = 1.$$

Since ρ takes non-negative functions to positive operators, $\psi(f)$ is non-negative whenever f is non-negative. By the Riesz Representation Theorem, there exists a unique non-negative Radon measure μ on the Borel subsets of $sp(T)$ such that

$$\psi(f) = \int_{sp(T)} f\, d\mu \quad \text{for all } f \text{ in } C(sp(T)).$$

We have $1 = \psi(1) = \int_{sp(T)} 1\, d\mu = \mu(sp(T))$, so μ is a probability measure on $sp(T)$.

Now define a mapping U from $C(sp(T))$ into $\mathcal{H}$ by

$$U(f) = \rho(f)x_0.$$

Since

$$\begin{aligned}\|U(f)\|^2 = \|\rho(f)x_0\|^2 &= \langle \rho(f)x_0, \rho(f)x_0 \rangle = \langle \rho(f)^*\rho(f)x_0, x_0 \rangle \\ &= \langle \rho(\bar{f}f)x_0, x_0 \rangle = \int_{sp(T)} \bar{f}f\, d\mu = \|f\|_2^2\end{aligned}$$

for every f in $C(sp(T))$, U is isometric on $C(sp(T))$ when $C(sp(T))$ has the norm from $L^2(\mu)$. Now $C(sp(T))$ is dense in $L^2(\mu)$, and U maps $C(sp(T))$

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onto a dense subspace of $\mathcal{H}$, since x_0 is a cyclic vector for T. It follows that U extends by continuity to an isometry of $L^2(\mu)$ onto $\mathcal{H}$. Furthermore, for any f and g in $C(sp(T))$,

$$(U^{-1}\rho(f)U)g = U^{-1}\rho(f)\rho(g)x_0 = U^{-1}\rho(fg)x_0 = fg.$$

Thus for any f in $C(sp(T))$, the operator $U^{-1}\rho(f)U$ agrees on $C(sp(T))$ with multiplication by f. Since $C(sp(T))$ is dense in $L^2(\mu)$ and multiplication by f is continuous from $L^2(\mu)$ into itself, $U^{-1}\rho(f)U$ agrees on all of $L^2(\mu)$ with multiplication by f, and the proof is complete. □

To extend this result to a general normal operator on $\mathcal{H}$ we need the notion of a direct sum of Hilbert spaces.

Definition 33 Let $\{\mathcal{H}_\gamma\}_{\gamma\in\Gamma}$ be a family of Hilbert spaces. Let $\mathcal{H}$ be the subset of the Cartesian product of the $\mathcal{H}_\gamma$ consisting of all those functions x on Γ that satisfy $\sum_\gamma \|x_\gamma\|^2 < +\infty$, and define $\langle x, y\rangle = \sum_\gamma \langle x_\gamma, y_\gamma\rangle$. The Hilbert space $\mathcal{H}$, with the inner product $\langle\ \rangle$, is the *Hilbert space direct sum* of the family $\{\mathcal{H}_\gamma\}_{\gamma\in\Gamma}$.

The reader should verify that the functional $\langle\ \rangle$ is an inner product and that the space $\mathcal{H}$ is a Hilbert space. We identify each $\mathcal{H}_\gamma$ with a closed subspace of $\mathcal{H}$, and we note that these closed subspaces are pairwise orthogonal in $\mathcal{H}$.

Returning now to a general Hilbert space $\mathcal{H}$, we observe that for each vector x in $\mathcal{H}$, the space $[C^*(T)x]$ is reducing for T, and the vector x is cyclic for the restriction of T to $[C^*(T)x]$. An application of Zorn's Lemma now establishes the following result.

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Proposition 56 *Let T be a normal element of $\mathcal{B}(\mathcal{H})$. Then there is a maximal family of a pairwise orthogonal closed subspaces of $\mathcal{H}$ that are reducing for T, each of which contains a vector that is cyclic for the restriction of T to that subspace. Moreover, $\mathcal{H}$ is isomorphic as a Hilbert space to the direct sum of the spaces in this maximal family.*

Theorem 49 *Let T be a normal element of $\mathcal{B}(\mathcal{H})$. Then there exist a measure space (X, μ) and an isometry U of $L^2(X, \mu)$ onto $\mathcal{H}$ such that $U^{-1}TU$ is a multiplication operator on $L^2(X, \mu)$.*

Sketch of proof: Write $\mathcal{H}$ as the direct sum of a maximal family of subspaces $\mathcal{H}_\gamma$ as in the proposition, and for each γ let T_γ be the restriction of T to $\mathcal{H}_\gamma$. For each γ construct a measure μ_γ and an isometry U_γ as in the theorem on normal operators with cyclic vectors. Let $f_\gamma(\lambda) = \lambda$ on $sp(T_\gamma)$, and let X be the disjoint union of the measure spaces $(sp(T_\gamma), \mu_\gamma)$. Define f on X by means of the f_γ, and define U on the direct sum of the spaces $L^2(sp(T_\gamma), \mu_\gamma)$ in the obvious way. Note that on each summand, T acts by $(Th)(\lambda) = \lambda h(\lambda)$, i.e. by multiplication by the identity function on $sp(T_\gamma)$. $\square$

We illustrate this result by returning to the case of the normal n by n matrix M and the corresponding operator T on $\mathbf{C}^n$.

For any eigenvalue λ of T, a vector v is an eigenvector for T and λ if and only if v is an eigenvector for T^* (since the kernels of the normal operators $T - \lambda I$ and $T^* - \bar{\lambda} I$ coincide), and the span of such a vector v is a one-dimensional reducing subspace for T. Thus choosing an orthonormal basis $\{v_1, \ldots, v_n\}$ for $\mathbf{C}^n$ consisting of eigenvectors of M amounts to choosing a maximal family of pairwise orthogonal reducing subspaces of $\mathbf{C}^n$, each

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containing a (unit) vector cyclic for T, as in the proposition above. For each of these reducing subspaces, the probability space $(sp(T_i), \mu_i)$ described in the proof of the last theorem has $sp(T_i) = \{\lambda_i\}$, where T_i is the restriction of T to the span of $\{v_i\}$ and λ_i is the eigenvalue of T corresponding to v_i. The disjoint union of these spaces is a set of n points, on which we have a purely atomic measure that gives each of the points measure one. The corresponding $L^2(X, \mu)$ is then just another copy of $\mathbf{C}^n$, on which T acts by multiplication by a vector $(\lambda_1, \ldots, \lambda_n)$ of eigenvalues, repeated as necessary according to their multiplicities in M. We may of course regard this vector of eigenvalues as a bounded measurable function on the measurable space (X, μ). Note that the inclusion of the multiplicities is essential, since the space $L^2(X, \mu)$ is required to have the same dimension as $\mathbf{C}^n$.

Definition 34 Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let S and T be bounded linear operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. We say that S and T are *unitarily equivalent* if there exists a linear isometry U of $\mathcal{H}$ onto $\mathcal{K}$ such that $S = U^{-1}TU$.

Note that if $\mathcal{H} = \mathcal{K}$, then U is actually a unitary operator on $\mathcal{H}$ and unitary equivalence is an equivalence relation on $\mathcal{B}(\mathcal{H})$.

Unitary equivalence is a rather strong type of equivalence. We might for example define S and T to be equivalent if they have the same spectrum, or perhaps if they have homeomorphic spectra. If S and T are unitarily equivalent, then they certainly have the same spectrum (exercise), so unitary equivalence implies either of these other relationships. Even for normal matrices, neither of these other notions of equivalence takes multiplicity into

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account, and so unitary equivalence does a better job of distinguishing operators.

The main result of this section is that every normal operator on $\mathcal{B}(\mathcal{H})$ is unitarily equivalent to multiplication by a bounded measurable function on an L^2-space. For an example with no eigenvalues, consider the mapping T on $L^2[0,1]$ given by $(Th)(\lambda) = \lambda h(\lambda)$. Note that since the constant function 1 in $L^2[0,1]$ is a cyclic vector for this T, all its spectral values have multiplicity one.

Exercises

1. Prove the proposition on reducing subspaces.

2. Verify that the functional $\langle\ \rangle$ in the definition of the direct sum is an inner product and that the space $\mathcal{H}$ in that definition is a Hilbert space.

3. Show that if S and T are unitarily equivalent, then they have the same spectrum.

4. Complete the details of the last theorem of this section.

5. Let T be the right shift on $l_2(\mathbf{Z})$. Find a cyclic vector for T and represent T as a multiplication operator on an $L^2(\mu)$.

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Commutation and Topological Closures: von Neumann's Double Commutant Theorem

Let T and N be elements of $\mathcal{B}(\mathcal{H})$, with N normal. We have seen that the spectral theorem, in its measurable calculus version, includes the assertion that an element W of $\mathcal{B}(\mathcal{H})$ commutes with N if and only if W commutes with every spectral projection $E(\omega)$ for N. In the last section we saw also that a closed subspace $\mathcal{K}$ of $\mathcal{H}$ is reducing for T if and only if the orthogonal projection $E_{\mathcal{K}}$ of $\mathcal{H}$ onto $\mathcal{K}$ commutes with T.

Definition 35 Let $\mathcal{S}$ be a subset of $\mathcal{B}(\mathcal{H})$. The commutant of $\mathcal{S}$ is the subset $\mathcal{S}'$ of $\mathcal{B}(\mathcal{H})$ given by

$$\mathcal{S}' = \{W \in \mathcal{B}(\mathcal{H}) : SW = WS \text{ for all } S \text{ in } \mathcal{S}\}.$$

Thus for normal N we have $\{N\}' = \{E(\omega) : \omega \text{ is a Borel subset of } sp(N)\}'$, and for any bounded linear T, $\mathcal{K}$ is reducing if and only if $E_{\mathcal{K}} \in \{T\}'$.

Exercise Let $\mathcal{S}$ be a subset of $\mathcal{B}(\mathcal{H})$. Then a) $\mathcal{S}'$ is a weak operator closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity operator I; b) $\mathcal{S}$ is stable under the adjoint operation (i.e., $S \in \mathcal{S}$ implies that $S^* \in \mathcal{S}$) if and only if $\mathcal{S}^* = \mathcal{S}$, where $\mathcal{S}^* = \{S^* : S \in \mathcal{S}\}$; c) if $\mathcal{S}$ is stable under the adjoint, then $\mathcal{S}'$ is a weak operator closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

Proposition 57 *Let $\mathcal{S}$ and $\mathcal{T}$ be subsets of $\mathcal{B}(\mathcal{H})$. Then a) $\mathcal{S} \subseteq \mathcal{S}''$, and $\mathcal{T}' \subseteq \mathcal{S}'$ whenever $\mathcal{S} \subseteq \mathcal{T}$; and b) $\mathcal{S}''' = \mathcal{S}'$.*

Proof: Part a) is obvious. For b), we have by a) that $\mathcal{S}''' \subseteq \mathcal{S}'$. If $T \in \mathcal{S}'$ and $W \in \mathcal{S}''$, then $TW = WT$. Thus $T \in \mathcal{S}'$ implies $T \in \mathcal{S}''$, so $\mathcal{S}' \subseteq \mathcal{S}'''$. □

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It follows immediately from the exercise above that the commutant of any $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is always a weak operator closed $*$-subalgebra of of $\mathcal{B}(\mathcal{H})$ containing the identity operator I. Since the strong operator topology contains the weak operator topology, any such commutant is also strong operator closed. The main result of this section shows that for a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing I, the strong operator closure is always a commutant (in fact the commutant of a commutant).

Theorem 50 (von Neumann's Double Commutant Theorem) *Let A be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity operator I on $\mathcal{H}$. Then the strong operator closure of A agrees with the commutant A'' of the commutant of A.*

To prove this theorem we need some observations about direct sums of Hilbert spaces and an easy lemma about invariant subspaces. The reader should verify the assertions in the next few paragraphs.

Let $\tilde{\mathcal{H}}$ be the Hilbert space direct sum $\tilde{\mathcal{H}} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ of n copies of $\mathcal{H}$. (See the previous section for a description of Hilbert space direct sums.) For any W in $\mathcal{B}(\mathcal{H})$, we let $\tilde{W}$ be the operator on $\tilde{\mathcal{H}}$ given by

$$\tilde{W}(y_1, y_2, \ldots, y_n) = (Wy_1, Wy_2, \ldots, Wy_n).$$

(We may choose on occasion to refer to the operator $\tilde{W}$ as the *n-fold direct sum* of W and to denote it by $W \oplus \cdots \oplus W$.) It is easy to check that $\tilde{W}$ is an element of $\mathcal{B}(\tilde{\mathcal{H}})$. Moreover, we can realize $\mathcal{B}(\tilde{\mathcal{H}})$ in the obvious way as the algebra $M_n(\mathcal{B}(\mathcal{H}))$ of all n by n matrices with entries from $\mathcal{B}(\mathcal{H})$. (That is, $\mathcal{B}(\tilde{\mathcal{H}})$ and $M_n(\mathcal{B}(\mathcal{H}))$ are naturally $*$-isomorphic.) Under this realization,

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the operator $\tilde{W}$ becomes the diagonal n by n matrix all of whose diagonal entries are W.

Now let A be any $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing I, and let $\tilde{A}$ be the algebra $\{\tilde{a} : a \in A\}$ of all diagonal matrices with a uniform diagonal entry that comes from A. It is easy to verify (exercise) that the commutant of $\tilde{A}$ is the algebra $M_n(A')$, i.e., the algebra of all n by n matrices with entries from the commutant A' of A, and that $(M_n(A'))' = \{\tilde{W} : W \in A''\}$.

Lemma 8 *Let A be an adjoint-stable subset of $\mathcal{B}(\mathcal{H})$. Then an orthogonal projection E in $\mathcal{B}(\mathcal{H})$ lies in A' if and only if the range of E is invariant under every element of A.*

Proof: E lies in A' if and only if E commutes with T and with T^* for every element of A, so E lies in A' if and only if the range of E is reducing for every element of A. Since A is adjoint-stable, this is equivalent to saying that the range of E is invariant under every element of A. □

Proof of Theorem 50: Since $A \subseteq A''$ and A'' is strong operator closed, the strong operator closure of A is contained in A''. Thus it suffices to show that A'' is contained in the strong operator closure of A.

Suppose then that $T \in A''$. We must show that every strong operator basic neighborhood of T contains an element of A. To this end, let $x_1, x_2, \ldots, x_n$ be vectors in $\mathcal{H}$, and let $\epsilon > 0$; we must find an element T_0 of A such that

$$\|(T - T_0)x_i\| < \epsilon$$

holds for all $i = 1, 2, \ldots, n$.

Now, as above, let $\tilde{\mathcal{H}}$ be the Hilbert space direct sum $\tilde{\mathcal{H}} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ of n copies of $\mathcal{H}$, and for any W in $\mathcal{B}(\mathcal{H})$, let $\tilde{W}$ be the operator on $\tilde{\mathcal{H}}$ given

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by

$$\tilde{W}(y_1, y_2, \ldots, y_n) = (Wy_1, Wy_2, \ldots, Wy_n).$$

We put $\tilde{x} = (x_1, x_2, \ldots, x_n)$, so that $\tilde{x} \in \tilde{\mathcal{H}}$, and we let $\tilde{A} = \{\tilde{a} : a \in A\}$. Then $\tilde{A}$ is a $*$-subalgebra of $\mathcal{B}(\tilde{\mathcal{H}})$, and the subspace $\tilde{A}\tilde{x} = \{\tilde{a}\tilde{x} : a \in A\}$ of $\tilde{\mathcal{H}}$ is invariant under each element of $\tilde{A}$. It follows that the norm-closure $[\tilde{A}\tilde{x}]$ of $\tilde{A}\tilde{x}$ is also invariant under each element of $\tilde{A}$. Thus the projection E of $\tilde{\mathcal{H}}$ onto $[\tilde{A}\tilde{x}]$ commutes with every element of $\tilde{A}$. That is, E is an element of the commutant $(\tilde{A})'$.

Now let us return to the original operator $T \in A''$. If we can show that $\tilde{T} \in (\tilde{A})''$, then $\tilde{T}$ will commute with E, and so the range of E will then be invariant under $\tilde{T}$. Then $\tilde{T}\tilde{x} = (Ty_1, Ty_2, \ldots, Ty_n)$ lies in this range, since $\tilde{x} = \tilde{I}\tilde{x} = (Ix_1, Ix_2, \ldots, Ix_n) \in E(\tilde{\mathcal{H}})$. Since $\tilde{A}\tilde{x}$ is dense in $[\tilde{A}\tilde{x}] = \text{Range}(E)$, there exists T_0 in A such that $\|\tilde{T}\tilde{x} - \tilde{T}_0\tilde{x}\|$ is arbitrarily small. This means in particular that we may choose T_0 in A such that $\|(T - T_0)x_i\| < \epsilon$ holds for all $i = 1, 2, \ldots, n$.

Finally, to see that $\tilde{T} \in (\tilde{A})''$, realize $\mathcal{B}(\tilde{\mathcal{H}})$ as $M_n(\mathcal{B}(\mathcal{H}))$ and $\tilde{T}$ as a diagonal matrix all of whose diagonal entries are T. Then the commutant of $\tilde{A}$ is $M_n(A')$, and its double commutant $(M_n(A'))'$ is $\{\tilde{W} : W \in A''\}$. Since $T \in A''$, we have $\tilde{T} \in (\tilde{A})''$ as desired, and the proof is complete. $\square$

Corollary 46 *Let A be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity operator I on $\mathcal{H}$. Then the following are equivalent:*

a) *A is weak operator closed;*

b) *A is strong operator closed;*

c) *$A = A''$.*

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Proof: It follows from the theorem that A is strong operator closed if and only if $A = A''$, so conditions b) and c) are equivalent. Since the strong operator topology contains the weak operator topology, condition a) implies condition b). It suffices then to show that condition c) implies condition a). But if $A = A''$, then A is a commutant and so is weak operator closed. □

Definition 36 A $*$-subalgebra A of $\mathcal{B}(\mathcal{H})$ containing the identity operator on $\mathcal{H}$ is called a *von Neumann algebra in* $\mathcal{H}$ if it satisfies (any one of) the three conditions of the corollary. If T is an element of $\mathcal{B}(\mathcal{H})$, then the double commutant of $C^*(T)$ is the *von Neumann algebra generated by* T.

The von Neumann algebra generated by T is thus the smallest von Neumann algebra in $\mathcal{B}(\mathcal{H})$ that contains T, and it coincides with both the weak and the strong operator closures of $C^*(T)$. Since the commutants of $C^*(T)$ and of $\{T, T^*\}$ coincide, the von Neumann algebra generated by T also agrees the weak and strong closures of the $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by T. When T is normal, this is just the weak operator closure of the algebra of all polynomials in T and T^*.

It follows from results in previous sections that for a normal operator T, the spectral projections of T all lie in the von Neumann algebra generated by T (and in fact generate the same von Neumann algebra as T). In particular, the projection onto the kernel and the projection onto the closure of the range of any normal T are elements of the von Neumann algebra generated by T.

Now let A be any von Neumann algebra in $\mathcal{B}(\mathcal{H})$, and let T be an element of A. If T is positive, then we may approximate T arbitrarily well in the norm

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topology by linear combinations of spectral projections for T. Since every element of A is a linear combination of four positive elements in A, it follows immediately that every von Neumann algebra A is the norm-closed linear span of the projections that it contains. Moreover, as the operators T^* and TT^* always have the same kernel, the closures of the ranges of T and TT^* must coincide. Thus for any von Neumann algebra A and any T in A, A must always contain the projections onto the kernel and the closure of the range of T, even if T is not normal. Thus von Neumann algebras are always very rich in projections. For comparison, note that even commutative C^*-algebras may fail to have any projections at all apart from zero and the identity. It follows also that when X is connected, no infinite-dimensional algebra of the form $C(X)$ can ever be $*$-isomorphic to a von Neumann algebra.

Exercises

1. Let $\mathcal{S}$ be a subset of $\mathcal{B}(\mathcal{H})$. Show that:

a) $\mathcal{S}'$ is a weak operator closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity operator I;

b) $\mathcal{S}$ is stable under the adjoint operation (i.e, $S \in \mathcal{S}$ implies that $S^* \in \mathcal{S}$) if and only if $\mathcal{S}^* = \mathcal{S}$, where $\mathcal{S}^* = \{S^* : S \in \mathcal{S}\}$; and

c) if $\mathcal{S}$ is stable under the adjoint, then $\mathcal{S}'$ is a weak operator closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

2. a) Verify the assertions above about $\tilde{\mathcal{H}} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ and $\mathcal{B}(\tilde{\mathcal{H}})$. In particular, show that there is a natural $*$-isomorphism between $\mathcal{B}(\tilde{\mathcal{H}})$

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and $M_n(\mathcal{B}(\mathcal{H}))$ that carries $\tilde{W} = W \oplus \cdots \oplus W$ to a diagonal matrix with a uniform diagonal entry W.

b) Then show that for any $*$-subalgebra A of $\mathcal{B}(\mathcal{H})$ containing I, we have

$$(\tilde{A})' = M_n(A') \quad \text{and} \quad (M_n(A'))' = \{\tilde{W} : W \in A''\}.$$

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