Real Analysis Exam

[1] For $\varepsilon > 0$ and $k > 0$, denote by $A(k, \varepsilon)$ the set of $x \in \mathbb{R}$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{k\,|q|^{2+\varepsilon}} \qquad \text{for any integers } p, q \text{ with } q \neq 0.$$

Show that $\mathbb{R} \setminus \bigcup_{k=1}^{\infty} A(k, \varepsilon)$ is of Lebesgue measure zero.

[2] Fix an enumeration of all rational numbers: $r_1, r_2, r_3, \cdots$. For $x \in \mathbb{R}$, define

$$f(x) = \text{the cardinal number of the set } \{n \ : \ |x - r_n| \leq \frac{1}{2^n}\}.$$

(a) Show that f is Lebesgue measurable.

(b) Evaluate $\displaystyle\int_{\mathbb{R}} f(x)dx$.

[3] Let X be a set and $\mathcal{M}$ a σ-algebra of subsets of X (i.e., $\emptyset$, $X \in \mathcal{M}$ and $\mathcal{M}$ is closed under taking complements and countable unions of sets in $\mathcal{M}$).

(a) If μ is an extended real valued function on $\mathcal{M}$, what conditions must μ satisfy in order to be called a *measure*?

(b) Take $X = \mathbb{R}^n$ and let $\mathcal{M}$ be the set of *all subsets* of $\mathbb{R}^n$. Is $\mathcal{M}$ a σ-algebra?

(c) With X and $\mathcal{M}$ as in (b) above, let $d \in [0, n]$ and define d-dimensional Hausdorff measure $\mathcal{H}^d : \mathcal{M} \to \mathbb{R}$ by

$$\mathcal{H}^d(A) = \lim_{r \searrow 0} \left(\inf \left\{ \sum_{j=1}^{\infty} [\operatorname{diam}(A_j)]^d : A \subset \cup_{j=1}^{\infty} A_j, \operatorname{diam}(A_j) \leq r \right\} \right). \qquad (1)$$

Here $\operatorname{diam}(A_j) = \sup\{\|x - y\| : x, y \in A_j\}$ is the diameter of A_j. Show that the limit in (1), and hence $\mathcal{H}^d$, is well defined.

(d) Is $\mathcal{H}^1$ a measure? Justify your answer.

[4] Let $f : \mathbb{R} \to \mathbb{R}$ be in $L^1(\mathbb{R})$, and let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function of period 1 and $\int_0^1 g(x)\, dx = 0$. Find

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) g(nx)\, dx.$$

Hint: You may use the fact that step functions are dense in $L^1(\mathbb{R})$.

[5] Let $f : [0,1] \to [0,1]$ be continuously differentiable and satisfy $f(0) = 0,\ f(1) = 1$.

(a) Show that the Lebesgue measure of

$$f\Big(\{x \in [0,1] : |f'(x)| < 1/m\}\Big)$$

is less than or equal to $1/m$.

(b) Use part (a) to show that there is at least one horizontal line $y = y_0 \in [0,1]$ which is nowhere tangent to the graph of f. Recall that the graph of f is $\{(x, f(x)) : x \in [0,1]\}$.

[6] Let X, Y, and Z be metric spaces and $f : X \to Y$ and $g : Y \to Z$ be maps. Assume further that

- X is compact;
- f is surjective and continuous; and
- $g \circ f$ is continuous.

Show that g is continuous.

[7] Let H be a real Hilbert space with norm $\| \ \|$ and inner product $\langle \ , \ \rangle$. Assume that $B : H \times H \to \mathbb{R}$ is bilinear (that is, $B(x,y)$ is linear in x for any fixed y and is linear in y for any fixed x). Assume further that there are positive constants C_1 and C_2 such that

$$\begin{aligned} |B(x,y)| &\leq C_1 \|x\| \, \|y\| \qquad x \in H, y \in H; \\ |B(x,x)| &\geq C_2 \|x\|^2 \qquad x \in H. \end{aligned}$$

(a) Show that there is a bounded linear operator $A : H \to H$ such that $B(x,y) = \langle Ax, y \rangle$ for all $x, y \in H$.

(b) Show that the operator A is one-to-one and onto.

[8] Let X be a complex Banach space, $I : X \to X$ denote the identity, and $S, T : X \to X$ be bounded linear operators. Denote by $\sigma(A) \subset \mathbb{C}$ the spectrum of operator A.

(a) Show that $I - ST$ has a bounded inverse if and only if $I - TS$ has a bounded inverse.

(b) Show that $\sigma(ST) \setminus \{0\} = \sigma(TS) \setminus \{0\}$.

(c) Show that $ST - TS \neq I$.

Algebra Exam

1. Let $n \geq 5$. Prove the following:

   (a) The only non-trivial normal subgroups of S_n is A_n.

   (b) S_n has no subgroup of index r, where $2 < r < n$.

   (c) List the normal subgroups of S_4.

2. (a) Let H be a proper subgroup of a finite group G. Show that G is not union of all conjugates of H.

   (b) Give an example of a group G, having a subgroup H, and an element a, such that $aHa^{-1} \subset H$, but $aHa^{-1} \neq H$.

3. A commutative ring A is called a Boolean ring if $x^2 = x$ for all $x \in A$.

   (a) Prove that if a Boolean ring contains no divisors of 0 it is either $\{0\}$ or is isomorphic to $\mathbb{Z}/(2)$. Deduce that in a Boolean ring every prime ideal is maximal.

   (b) Prove that in a Boolean ring every ideal $I \neq A$ is the intersection of the prime ideals containing I.

4. (a) Let R be a commutative ring with identity. Prove that every proper ideal I of R is contained in some maximal proper ideal.

   (b) Let k be a field, $R = k[x, y]$ and $I = (x^2 + y^2 - 1)$. Exhibit a maximal proper ideal containing I. Prove your claim.

5. Let f be a polynomial of degree n with coefficients in a field k of characteristic 0.

   (a) What is meant by a splitting field of f ?

   (b) Let L be a splitting field of f over k. Prove that $[L : k]$ is a divisor of $n!$.

6. Let F_q denote the finite field with q elements. For a prime p, consider the field F_{p^n} containing F_p as a subfield.

   (a) Prove that the group of automorphisms of F_{p^n} is cyclic of order n.

   (b) What is meant by a separable field extension ?

   (c) What is meant by a normal field extension ?

   (d) Is the field extension F_{p^n} over F_p separable and/or normal ?

7. Prove that a real quadratic form $Q(X_1, \ldots, X_n)$ can always be reduced to the form, $Q(X_1, \ldots, X_n) = \lambda_1 X_1^2 + \cdots + \lambda_n X_n^2$, with $\lambda_i \in \mathbb{R}$, using a linear change in co-ordinates.

8. Recall that $SL(n, \mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) | \det(A) = 1\}$ and $sl(n, \mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) | \operatorname{tr}(A) = 0\}$. Prove that, $\exp(tA) \in SL(n, \mathbb{R})$ for all $t \in \mathbb{R}$ if and only if $A \in sl(n, \mathbb{R})$.