Algebra Comprehensive Exam
Fall 2004

Instructions: Attempt any five questions, and please provide careful and complete answers. If you attempt more questions, specify which five should be graded.

1. (a) Prove that a group of order $1225 = 7^2 \cdot 5^2$ is abelian.

   (b) List the groups of order 1225 up to isomorphism.

2. Let G be a group with identity element e, with the property that for any two elements $x, y \in G \setminus \{e\}$, there exists an automorphism σ of G with $\sigma(x) = y$.

   (a) Prove that all elements of $G \setminus \{e\}$ have the same order.

   (b) If G is finite, prove that it is abelian.

3. Let $GL_n(\mathbb{C})$ be the multiplicative group of $n \times n$ matrices of complex numbers. Prove that every element of $GL_n(\mathbb{C})$ of finite order is diagonalizable.

4. Determine all maximal ideals of the ring

$$\mathbb{Z}[x]/(120, x^3 + 1).$$

5. For which integers $n \geq 1$ does the polynomial

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \in \mathbb{Q}[x]$$

   have multiple roots?

6. For an integer $n \geq 3$, consider a regular n-sided polygon inscribed in a circle of radius 1. Let $P_1, \ldots, P_n$ be its vertices, and λ_k be the length of the line joining P_n and P_k for $1 \leq k \leq n-1$. Prove that

$$\lambda_1 \cdots \lambda_{n-1} = n.$$

7. Let A be a real $n \times n$ matrix and let

$$M = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\},$$

   where $|\lambda|$ denotes the absolute value of the complex number λ.

   (a) If A is symmetric, prove that $\|Ax\| \leq M\|x\|$ for all $x \in \mathbb{R}^n$, where $\| \;\; \|$ denotes the Euclidean norm on $\mathbb{R}^n$.

   (b) Is this true if A is not symmetric? Prove or disprove.