Comprehensive Exam, Fall 2004 (Analysis)

Problem 1: Prove or give a counterexample to the following statement: Every function $f : [0, +\infty) \to \mathbb{R}$ for which the improper Riemann integral

$$\int_0^\infty f(x)dx$$

is convergent is Lebesgue integrable on $[0, +\infty)$.

Problem 2: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n : \Omega \to \mathbb{R}, n \geq 1$, and $g : \Omega \to \mathbb{R}$ be functions in $L^1(\mu)$ such that there exists a constant $C > 0$ such that

$$\int_\Omega |f_n| \, d\mu \leq C$$

for all $n \geq 1$. Suppose moreover that

$$\frac{1}{n} f_n^2 \leq g \quad \text{on } \Omega.$$

Show that

$$\int_\Omega \frac{1}{n} f_n^2 \, d\mu \to 0 \quad \text{as } n \to \infty.$$

Problem 3: Define

$$B = C([0,1]) = \{f : [0,1] \to \mathbb{R} : \text{ f is continuous}\}, \quad \|f\|_B = \max_{0 \leq x \leq 1} |f(x)|$$

$$C = C^\alpha([0,1]) = \{f : [0,1] \to \mathbb{R} : f \in B \text{ and } \|f\|_C = \|f\|_B + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < +\infty\},$$

for some $\alpha \in (0,1]$. It is well known that equipped with the norms $\|\cdot\|_B$, and $\|\cdot\|_C$, the spaces B, and C respectively are Banach spaces (normed vector spaces complete with respect to the norm metric). Determine if the unit ball is compact in the spaces B and C. Is the unit ball of C compact as a subset of B?

Problem 4: Let $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ be a function such that
(i) for each $t \in [a,b]$, the function $x \to f(t,x)$ is continuous,
(ii) for each $x \in \mathbb{R}^n$, the function $t \to f(t,x)$ is Lebesgue measurable.
Show that f is $\mathcal{L} \otimes \mathcal{B}$ measurable, where $\mathcal{L}$ is the class of Lebesgue measurable sets on $[a,b]$, $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$, and $\mathcal{L} \otimes \mathcal{B}$ is the product σ-algebra of $\mathcal{L}$ and $\mathcal{B}$.

Problem 5: Fix a prime number p. A rational number x can be represented by $x = p^\alpha \frac{k}{l}$ with k, l not divisible by p, and $\alpha \in \mathbb{Z}$ is defined uniquely. Define $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ by

$$|x|_p := p^{-\alpha}, \quad \text{and} \qquad |0|_p := 0.$$

(a) Show that $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ and conclude that $d(x,y) := |x-y|_p$ defines a metric on $\mathbb{Q}$.

(b) Show that in the completion of $\mathbb{Q}$ w.r.t. the above metric a series of rational numbers

$$\sum_{n\geq 0} a_n$$

converges if and only if $|a_n|_p \to 0$.

Problem 6: Let (X,d) be a compact metric space and denote by $B_R(a) \subset X$ the closed ball of radius $R > 0$ centered at $a \in X$. Suppose μ is a positive Borel measure on X satisfying for some $\beta > 0$ and for all $r \in (0,1)$ and $a \in X$

$$c_1\, r^\beta \leq \mu(B_r(a)) \leq \; r^\beta,$$

with $c_1 > 0$ independent of r and a. Fix a point $a \in X$. Find all $\alpha \in \mathbb{R}$ for which $x \mapsto d(x,a)^\alpha$ is in $L^1(X, d\mu)$.

Problem 7: Let H be a Hilbert space. Show that if $T : H \to H$ is symmetric, i.e. $\langle x, Ty\rangle = \langle Tx, y\rangle$ for all $x, y \in H$, then T is linear and continuous.